Fluid Mechanics 4 (Mechanical)
Tutorial week 6

Question 1

Viscous oil flows steadily between two horizontal, stationary, parallel plates, as shown in Fig. 1. The flow between the plates is fully developed and steady. The gap width between the plates is $h = 3$ mm. The oil viscosity is 0.5 Ns/m^2 and the pressure gradient is -1200 Pa/m.

- Determine the magnitude of shear stress acting on the top and bottom plates.
- Determine the volume flow rate per unit width between the plates. (The volume flow rate is the volume of fluid which passes per unit time.)

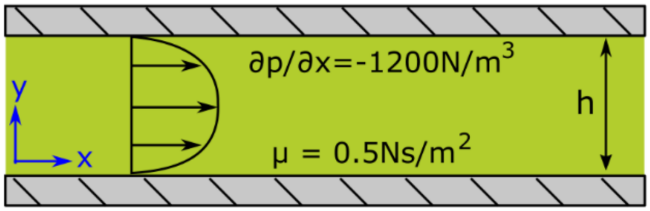


Figure 1: Oil flow in a channel

Solution

Shear Stress on the Plates

For fully developed flow between stationary plates, the velocity distribution is parabolic. The shear stress τ is related to the gradient of the velocity:

$$\tau = \mu \left(\frac{du}{dy} \right)$$

where $u(y)$ is the velocity profile. The velocity profile for Poiseuille flow is given by:

$$u(y) = -\frac{1}{2\mu} \left(\frac{dP}{dx} \right) y(h - y)$$

We need to compute $\frac{du}{dy}$ to find the shear stress at $y = 0$ (bottom plate) and $y = h$ (top plate). Taking the derivative of $u(y)$ with respect to y:

$$\frac{du}{dy} = -\frac{1}{2\mu} \left(\frac{dP}{dx} \right) (h - 2y)$$

At $y = 0$ (the bottom plate):

$$\left. \frac{du}{dy} \right|_{y=0} = -\frac{1}{2\mu} \left(\frac{dP}{dx} \right) h$$

At $y = h$ (the top plate):

$$\left.\frac{du}{dy}\right|_{y=h} = \frac{1}{2\mu}\left(\frac{dP}{dx}\right) h$$

The shear stress at the plates is:

$$\tau_{\text{bottom}} = -\mu \cdot \frac{1}{2\mu}\left(\frac{dP}{dx}\right) h = -\frac{1}{2}\left(\frac{dP}{dx}\right) h$$

$$\tau_{\text{top}} = \frac{1}{2}\left(\frac{dP}{dx}\right) h$$

Now, substituting the given values:

$$\tau_{\text{bottom}} = 1.8\,\text{Pa}$$

$$\tau_{\text{top}} = -1.8\,\text{Pa}$$

Thus, the magnitude of the shear stress on the top and bottom plates is 1.8 Pa.

Volume Flow Rate per Unit Width

The velocity profile for Poiseuille flow between two parallel plates is:

$$u(y) = -\frac{1}{2\mu}\left(\frac{dP}{dx}\right) y(h - y)$$

The total volume flow rate per unit width, q, is obtained by integrating the velocity profile over the gap width h:

$$q = \int_0^h u(y)\, dy$$

Substitute the velocity profile into the integral:

$$q = \int_0^h -\frac{1}{2\mu}\left(\frac{dP}{dx}\right) y(h - y)\, dy$$

Simplify the expression:

$$q = -\frac{1}{2\mu}\left(\frac{dP}{dx}\right) \int_0^h \left(hy - y^2\right)\, dy = -\frac{1}{2\mu}\left(\frac{dP}{dx}\right)\left(\frac{h^3}{2} - \frac{h^3}{3}\right) = -\frac{h^3}{12\mu}\left(\frac{dP}{dx}\right)$$

Substitute the given values:

$$q = 5.4 \times 10^{-6}\,\text{m}^2/\text{s}$$

Question 2

An incompressible, viscous fluid is placed between two horizontal, infinitely long flat plates (see Fig. 2). The two plates move in opposite directions with constant velocities U_1 and U_2. The fluid flow between the plates is considered to be steady. The pressure gradient in the x-direction is zero, and the only body force is due to the fluid weight. Derive an expression for the velocity distribution $u(y)$ between the two plates.

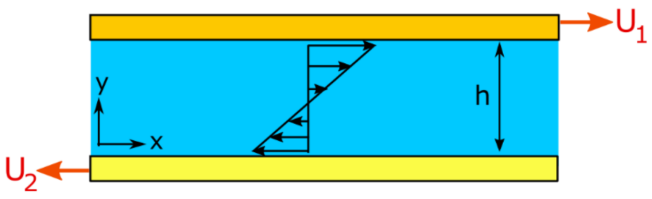


Figure 2: Flow between two plates.

Solution

Assumptions:

1. Incompressible flow, so the continuity equation gives:
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$
2. Velocity components v and w are assumed to be negligible. The primary motion is in the x-direction, and only horizontal information is given. This assumption appears suitable, but we should check its validity as we derive the governing equations.
3. The flow is steady, $\partial u/\partial t = 0$.
4. The plates are infinitely long, such that the flow is not changing in the z-direction.
5. Gravity only acts in the vertical direction.

Governing equations of motion
Continuity:
$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0$$
For incompressible flow:
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$
If v and w are negligible, the continuity equation simplifies to:
$$\frac{\partial u}{\partial x} = 0$$
This implies that u is not changing in the x-direction.
X-momentum equation:
$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \mu\frac{1}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$
then
$$\rho\left(0 + u\cdot 0 + 0\cdot\frac{\partial u}{\partial y} + 0\cdot 0\right) = 0 - 0 + \mu\left(0 + \frac{\partial^2 u}{\partial y^2} + 0\right)$$
This reduces to:
$$\frac{\partial^2 u}{\partial y^2} = 0$$
Integrating with respect to y:
$$\frac{\partial u}{\partial y} = C_1$$
where C_1 is a constant of integration. Integrating again with respect to y:
$$u = C_1 y + C_2$$
where C_2 is another constant of integration.
Applying boundary conditions:
At $y = 0$, the velocity is $-U_2$, so:
$$u = -U_2 \quad \Rightarrow \quad C_2 = -U_2$$
At $y = h$, the velocity is U_1, so:
$$u = U_1 \quad \Rightarrow \quad U_1 = C_1 h - U_2$$
Solving for C_1:
$$C_1 = \frac{U_1 + U_2}{h}$$
Thus, the velocity profile is:
$$u = \left(\frac{U_1 + U_2}{h}\right) y - U_2$$
The application of the y- and z-momentum equations will show that the y-pressure gradient is related to the hydrostatic pressure (i.e., due to the weight of fluid due to gravity), while there is no z-pressure gradient.

Question 3

The stream function for an incompressible, two-dimensional flow field is given as:

$$\psi = ay^2 - bx$$

Where a and b are constants. Is this flow irrotational? Can you determine the velocity potential ϕ?

Solution

Irrotational flow for a 2D flow in Cartesian coordinates: $\omega_z = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$

- $u = \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}(ay^2 - bx) = 2ay$
- $v = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}(ay^2 - bx) = b$
- $\omega_z = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \left(\frac{\partial}{\partial x}(b) - \frac{\partial}{\partial y}(2ay)\right) = 2a$

No, the flow is not irrotational. As such, you cannot determine the velocity potential.

Question 4

In planar polar coordinates the velocity can be expressed in the following way:

$$v_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \quad \text{and} \quad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r},$$

The velocity components in polar coordinates for an incompressible, planar flow are:

$$v_r = Ar^{-1} + Br^{-2}\cos\theta$$

$$v_\theta = Br^{-2}\sin\theta$$

Where A and B are constants. Determine the corresponding stream function.

Solution

The definition of the stream function in polar coordinates is:

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \quad \text{and} \quad v_\theta = -\frac{\partial \psi}{\partial r}$$

Thus:

$$\frac{1}{r}\frac{\partial \psi}{\partial \theta} = Ar^{-1} + Br^{-2}\cos\theta$$

$$\frac{\partial \psi}{\partial r} = -Br^{-2}\sin\theta$$

Integrate the above equations:

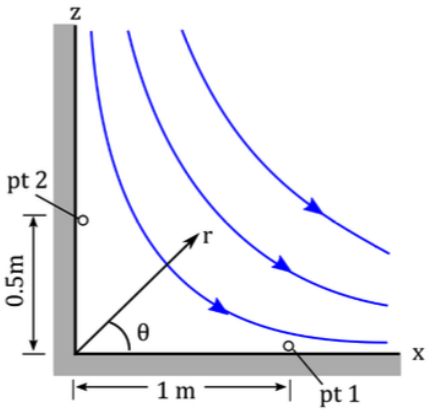
- $\psi = \int (A + Br^{-1}\cos\theta)\, d\theta = A\theta + Br^{-1}\sin\theta + C_1(r)$, where $C_1(r)$ is a constant of integration and could be a function of r.
- $\psi = \int (-Br^{-2}\sin\theta)\, dr = Br^{-1}\sin\theta + C_2(\theta)$, where $C_2(\theta)$ is a constant of integration and could be a function of θ. From the first equation, it is clear that $C_2(\theta) = A\theta$.
- To satisfy the two equations above, $\psi = A\theta + Br^{-1}\sin\theta$.

Question 5

A two-dimensional inviscid, incompressible fluid that flows in the vicinity of a 90° corner is described by the stream function:

$$\psi = 2r^2 \sin 2\theta$$

Where ψ has units of m^2/s and r has units of m. The flow in the x-z plane is horizontal (this is not a vertical flow).



(a) Determine the corresponding velocity potential function ϕ.

(b) Determine the pressure at point 2 near the wall if the pressure at point 1 is 30 kPa. Assume that the fluid density is $1000\,kg/m^3$ and that there is no elevation difference between points (1) and (2).

Solution

(a) The radial and azimuthal velocity components can be obtained from the stream function.

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial \theta}(2r^2 \sin 2\theta) = 4r \cos 2\theta$$

$$v_\theta = -\frac{\partial \psi}{\partial r} = -\frac{\partial}{\partial r}(2r^2 \sin 2\theta) = -4r \sin 2\theta$$

The velocity potential can be determined from the relationships: $v_r = \frac{\partial \phi}{\partial r}$ and $v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$

$$v_r = \frac{\partial \phi}{\partial r} = 4r \cos 2\theta$$

Integrate with respect to r: $\phi = \int (4r \cos 2\theta)\, dr = 2r^2 \cos 2\theta + C_1(\theta)$, where $C_1(\theta)$ is a constant of integration and could be a function of θ.

$$v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial \theta}(-4r \sin 2\theta)$$

Integrate with respect to θ: $\phi = \int (-4r^2 \sin 2\theta)\, d\theta = 2r^2 \cos 2\theta + C_2(r)$, where $C_2(r)$ is a constant of integration and could be a function of r.
To satisfy both expressions for ϕ, the velocity potential must have the form:

$$\phi = 2r^2 \cos 2\theta + C$$

Where C is an arbitrary constant. It is customary to let $C = 0$. Thus, the velocity potential for this corner flow is:

$$\phi = 2r^2 \cos 2\theta$$

(b) Since the velocity potential exists, the flow is irrotational in addition to being 2D, incompressible, and inviscid. Thus, we can use the Bernoulli equation to determine pressure and velocity differences between two points.

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

Since the elevation change between points 1 and 2 is zero, we can solve for an expression for p_2.

$$p_2 = p_1 + \frac{\rho}{2}(V_1^2 - V_2^2)$$

The challenge now is to determine the velocities at points 1 and 2. The velocity magnitude can be expressed in terms of its velocity components:

$$V^2 = v_r^2 + v_\theta^2$$

Thus:

$$V^2 = (4r\cos 2\theta)^2 + (-4r\sin 2\theta)^2$$

$$V^2 = 16r^2(\cos^2 2\theta + \sin^2 2\theta) = 16r^2$$

At point 1, $r = 1\,\text{m}$, while at point 2, $r = 0.5\,\text{m}$. Thus,

$$V_1^2 = 16 \times (1\,\text{m})^2 = 16\,\frac{\text{m}^2}{\text{s}^2}$$

$$V_2^2 = 16 \times (0.5\,\text{m})^2 = 4\,\frac{\text{m}^2}{\text{s}^2}$$

Substitute these values back into the expression for p_2:

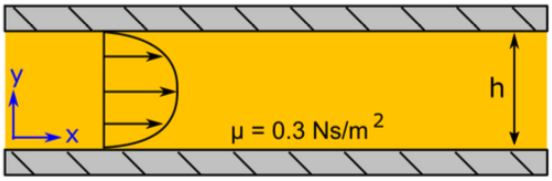
$$p_2 = p_1 + \frac{\rho}{2}(V_1^2 - V_2^2) = 30 \times 10^3 \frac{\text{N}}{\text{m}^2} \times \frac{1}{2} \times 1000 \frac{\text{kg}}{\text{m}^3}(16\frac{\text{m}^2}{\text{s}^2} - 4\frac{\text{m}^2}{\text{s}^2}) = 36\ \text{kPa}$$

Question 6

Two fixed, horizontal, parallel plates are spaced 4 mm apart. A viscous liquid with $\mu = 0.3\,\text{Ns/m}^2$ flows between the plates with a mean velocity of $u = 0.2\,\text{m/s}$.

(a) If the mean velocity between the plates is expressed as $\overline{u} = \frac{q}{h}$, where $q = \int_0^h u(y)\,dy$ is the volume flow rate per unit width, determine the pressure drop per unit length (in kPa/m) in the direction of the flow.

(b) Determine the maximum velocity (in m/s) within the channel.



Solution

Assumptions:

- The fluid is steady.
- Fluid is incompressible.
- The plate spacing is much smaller than the width of the plates, so we can ignore effects in the z-direction.
- Assume that the main velocity component is in the x-direction (i.e., $v = w = 0$).

(a) Write governing equations of interest to determine an expression for the velocity profile. From this velocity profile, we can solve for the volume flow rate and ultimately the pressure drop.

Continuity for incompressible, 2D flow:

- $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \rightarrow v = w = 0 \rightarrow \frac{\partial u}{\partial x} = 0$

x-momentum equation for incompressible flow:

- $\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$

- Removing negligible terms: $v = w = 0$, gravity acts only in the y direction, $\frac{\partial u}{\partial x} = 0 \rightarrow \frac{\partial^2 u}{\partial x^2} = 0$, steady flow.
- $0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} \rightarrow \frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\frac{\partial p}{\partial x}$
- Integrate twice to obtain expression of velocity profile:
- $\frac{\partial u}{\partial y} = \frac{1}{\mu}\frac{\partial p}{\partial x} y + C_1 \rightarrow u(y) = \frac{1}{2\mu}\frac{\partial p}{\partial x} y^2 + C_1 y + C_2$
- Apply boundary conditions to determine constants of integration:
- At $y = 0$, $u = 0 \rightarrow C_2 = 0$; at $y = h$, $u = 0 \rightarrow C_1 = -\frac{1}{2\mu}\frac{\partial p}{\partial x} h$
- $u(y) = \frac{1}{2\mu}\frac{\partial p}{\partial x}(y^2 - yh)$

Determine the pressure drop per unit width:

- $q = \int_0^h \left(\frac{1}{\mu}\frac{\partial p}{\partial x}(y^2 - yh)\right) dy = \frac{1}{2\mu}\frac{\partial p}{\partial x}\left(\frac{h^3}{3} - \frac{h^3}{2}\right) = -\frac{1}{12\mu}\frac{\partial p}{\partial x} h^3$
- $-\frac{1}{12\mu}\frac{\partial p}{\partial x} h^3 = q = uh$
- $\frac{\partial p}{\partial x} = -\frac{12\mu\overline{u}}{h^2} = -\frac{12 \times 0.3\,\frac{\text{Ns}}{\text{m}^2} \times 0.2\,\frac{\text{m}}{\text{s}}}{(0.004\,\text{m})^2} = -45,000\,\frac{\text{Pa}}{\text{m}} = -45\,\frac{\text{kPa}}{\text{m}}$

(b) The maximum velocity of the fluid occurs at $y = h/2$:

- $u_{\max} = \frac{1}{2\mu}\frac{\partial p}{\partial x}\left(\left(\frac{h}{2}\right)^2 - \frac{h^2}{2}\right) = -\frac{1}{8\mu}\frac{\partial p}{\partial x} h^2 = -\frac{1}{8 \times 0.3\,\frac{\text{Ns}}{\text{m}^2}} \times (-45000\,\frac{\text{N}}{\text{m}^3}) \times (0.004\,\text{m})^2 = 0.3\,\text{m/s}$

Exam-style questions

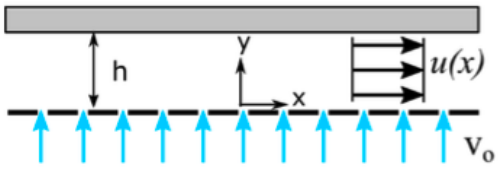
Question 7

Air flows steadily between two closely spaced parallel plates separated by height h by entering through a porous surface on the bottom plate. Assume that the viscous effects are negligible such that the horizontal velocity is uniform in the y direction, but varies as a function of x. Assume that the air is incompressible.

(a) Using a control volume, determine an expression for $u(x)$.

(b) Determine an expression for the y-velocity component $v(y)$ in the narrow passage.

(c) Determine an expression of the acceleration components of the fluid in between the narrow passage.



Solution

Assumptions:

- The fluid is steady.
- Fluid is incompressible.
- The flow entering the passage is uniform.
- Viscous effects can be neglected.

- The flow is 2D and variations in z can be neglected.

(a) Draw a control volume in between the plates. The flow entering the CV is from the vertical velocity (v_0) through the porous plate, while the velocity exiting is $u(x)$.

Apply continuity in the control volume: $0 = \frac{\partial}{\partial t}\int_{CV} \rho\, dV + \int_{CS} \rho \vec{V} \cdot dA$

- The first term is zero because the flow is steady.
- $\int_{CS} \rho \vec{V} dA$is the net fluid leaving the CV; i.e., fluid exiting - fluid entering = zero.
- $\rho u(x) hb - \rho v_0 x b = 0$, where b is the depth of the plates into the page.
- $u(x) = v_0 \left(\frac{x}{h}\right)$

(b) Apply the differential form of the continuity. The flow is 2D and incompressible

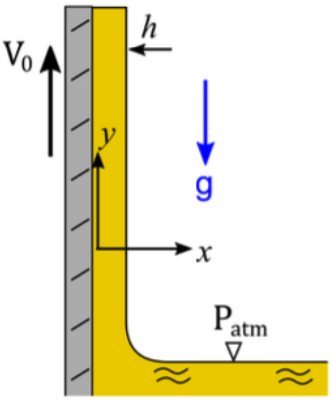
- $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$
- $\frac{\partial}{\partial x}\left(v_0 \frac{x}{h}\right) + \frac{\partial v}{\partial y} = 0 \rightarrow \frac{\partial v}{\partial y} = -\frac{v_0}{h}$
- Integrate the above expression with respect to y: $\partial v = \int_0^y -\frac{v_0}{h}\, dy \rightarrow v - v_0 = -\frac{v_0}{h} y + f(x)$
- In the above expression, $f(x)$ is a constant that is a function of x. However, since $v = v_0$, which is constant along $y = 0$, then $f(x) = 0$.
- Thus, $v = v_0\left(1 - \frac{y}{h}\right)$

(c) Acceleration: $\vec{a} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$ or $\vec{a_i} = \frac{\partial \vec{u}_i}{\partial t} + u\frac{\partial \vec{u}_i}{\partial x_i}$

- steady $\rightarrow \frac{\partial}{\partial t} = 0$; 2D, no z-velocity: $w = 0$ & $\frac{\partial}{\partial z} = 0$
- x acceleration: $a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = v_0 \frac{x}{h}\frac{\partial}{\partial x}\left(v_0 \frac{x}{h}\right) + v_0\left(1 - \frac{y}{h}\right)\frac{\partial}{\partial y}(v_0 \frac{x}{h})$
- $a_x = v_0^2 \frac{x}{h^2} + v_0(1 - \frac{y}{h}) \cdot 0 \rightarrow a_x = v_0^2 \frac{x}{h^2}$
- y acceleration: $a_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = v_0 \frac{x}{h}\frac{\partial}{\partial x}(v_0\left(1 - \frac{y}{h}\right)) + v_0\left(1 - \frac{y}{h}\right)\frac{\partial}{\partial y}(v_0\left(1 - \frac{y}{h}\right))$
- $a_y = v_0 \frac{x}{h} \cdot 0 + v_0(1 - \frac{y}{h})(-\frac{v_0}{h}) = \frac{v_0^2}{h}\left(\frac{y}{h} - 1\right)$
- $\vec{a} = v_0^2 \frac{x}{h^2}\hat{i} + \frac{v_0^2}{h}\left(\frac{y}{h} - 1\right)\hat{j}$

Question 8

A wide moving belt passes through a container of viscous fluid. The belt moves vertically upward with a constant velocity, V_0. Due to the viscous forces, the belt picks up a liquid film of thickness h. Gravity tends to make the fluid flow down the belt. Use the governing equations of fluid mechanics to determine:



(a) An expression of the velocity $v(x)$.

(b) The shear stress acting at the interface between the belt and fluid if the viscosity and density of the fluid is $\mu = 0.095\,\mathrm{Ns/m^2}$ and $\rho = 860\,\mathrm{kg/m^3}$ and the thickness of the fluid is $h = 3\,\mathrm{mm}$.

(c) The volume flow rate per unit width $q = \int v(x)\, dx$ if the belt velocity is $V_0 = 2\,\mathrm{m/s}$.

Solution

Assumptions:

- The fluid is steady.
- Fluid is incompressible.
- The fluid is open to the atmosphere. Since the pressure on the surface of the film is atmospheric, the pressure throughout the film must be atmospheric, i.e., $\frac{\partial p}{\partial x_i} = 0$.
- Assume that the film height is much smaller than the width of the belt, so we can ignore effects in the z-direction.
- Assume that the main velocity component is in the y-direction (i.e., $u = w = 0$).

a) Write governing equations of interest:

Continuity for incompressible, 2D flow:

- $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \rightarrow \frac{\partial v}{\partial y} = 0$

Y momentum equation:

- $\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = \rho g y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + \mu\frac{1}{3}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$
- $\rho(0 + 0 \cdot \frac{\partial v}{\partial x} + v \cdot 0 + 0) = \rho g y - 0 + \mu\left(\frac{\partial^2 v}{\partial x^2} + 0 + 0\right) + \mu\frac{1}{3}\frac{\partial}{\partial y}(0)$
- $0 = -\rho g + \mu\frac{\partial^2 v}{\partial x^2}$; the negative sign is used because gravity acts in the negative y direction
- $\frac{\partial^2 v}{\partial x^2} = \frac{\rho g}{\mu}$
- Integrate with respect to x: $\frac{\partial v}{\partial x} = \frac{\rho g}{\mu}x + C_1$
- Integrate with respect to x: $v(x) = \frac{\rho g}{2\mu}x^2 + C_1 x + C_2$
- Apply boundary conditions to determine constants of integration:
- At $x = 0$, $v = V_0$: $C_2 = V_0$
- At $x = h$, shear stress is zero (free surface) $\tau_{xy} = \mu\frac{\partial v}{\partial x} = 0$ or $\frac{\partial v}{\partial x} = 0 \rightarrow\ C_1 = -\frac{\rho g h}{\mu}$
- $v(x) = \frac{\rho g}{\mu}\left(\frac{x^2}{2} - xh\right) + V_0$

b) Shear stress acting at belt/fluid interface:

- $\tau_{xy} = \mu\frac{\partial v}{\partial x} = \mu(\frac{\rho g}{\mu}x - \frac{\rho g h}{\mu} = \rho g(x - h) = 860\frac{\text{kg}}{\text{m}^3} \times 9.81\frac{\text{m}}{\text{s}^2} \times (0 - 0.003\,\text{m}) = -25.3\,\frac{\text{N}}{\text{m}^2}$

c) Volume flow rate per unit width: $q = \int v(x)\,dx$

- $q = \int_0^h \left(\frac{\rho g}{\mu}\left(\frac{x^2}{2} - xh\right) + V_0\right)\,dx = \frac{\rho g}{\mu}\left(\frac{h^3}{6} - \frac{h^3}{2}\right) + V_0 h$
- $q = V_0 h - \frac{\rho g h^3}{3\mu} = 2\,\frac{\text{m}}{\text{s}} \times 0.003\,\text{m} - 860\frac{\text{kg}}{\text{m}^3} \times 9.81\frac{\text{m}}{\text{s}^2} v \times (0.003\text{m})^3/(3 \times 0.095\frac{\text{Ns}}{\text{m}^2})$
- $q = 0.0052\,\text{m}^2/\text{s}$

Question 9

A thin film of water with constant height h flows steadily down an inclined roof after a rainfall (see Fig. 3). In this thin layer of water, viscous effects are important to describe the fluid motion. The top of the film is exposed to the air (i.e., open atmosphere), such that the interface between the film and the air is considered to be a free surface. The film height relative to the roof dimensions is such that the roof can be assumed to be an infinitely long plate.

1. Using logical reasoning, determine the pressure gradient along the direction of the flow (i.e., $\partial p/\partial x$) for this thin film with free surface and shallow angle α. (Tip: don't use equations here, just make a reasonable assumption.)

2. If the primary flow direction is in the x-direction, determine an expression for $u(y)$.

3. Determine the volume flow rate per unit width (q, defined below) if the roof is inclined with $\alpha = 15°$ and $h = 1\,\text{mm}$. The properties of water are: $\rho = 1000\,\text{kg/m}^3$ and $\mu = 0.00152\,\text{Ns/m}^2$.

$$q = \int u(y)\, dy$$

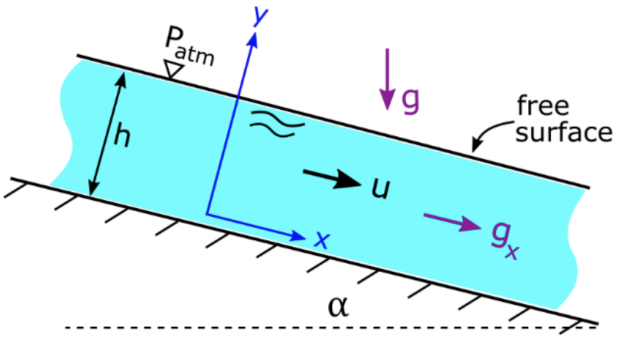


Figure 3: Film of water on a roof.

Solution

1. At the free surface, there is no shear (i.e., no velocity gradient at the interface). With a thin film, shallow angle, and free surface, the pressure gradient in the x-direction can easily be considered to be zero:

$$\frac{\partial p}{\partial x} = 0$$

Such an observation is often not explicitly asked in many fluid mechanic problems, but it is implicitly expected for the student to state this conclusion as a primary assumption to solve the problem.

2. List the governing assumptions:

- Free interface and shallow angle suggest that this flow is not pressure driven in the x-direction: $\frac{\partial p}{\partial x} = 0$
- Primary flow is in the x-direction: $v = w = 0$
- Water (liquid) is incompressible
- Flow is steady, $\frac{\partial u}{\partial t} = 0$

Write governing equations of interest:

Continuity:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0$$

For incompressible flow:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

If v and w are negligible, the continuity equation simplifies to:

$$\frac{\partial u}{\partial x} = 0$$

This implies that u is not changing in the x-direction.

X-momentum:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \mu\frac{1}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

Simplifying, as u, v, and w are zero:

$$\rho g_x + \mu\frac{\partial^2 u}{\partial y^2} = 0$$

Substituting $g_x = g\sin(\alpha)$:

$$0 = \rho g\sin(\alpha) + \mu\frac{\partial^2 u}{\partial y^2}$$

Simplifying further:

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\rho g\sin(\alpha)}{\mu}$$

Now, integrate with respect to y:

$$\frac{\partial u}{\partial y} = -\frac{\rho g\sin(\alpha)}{\mu}y + C_1$$

where C_1 is a constant of integration. Integrating again with respect to y:

$$u(y) = -\frac{\rho g\sin(\alpha)}{2\mu}y^2 + C_1 y + C_2$$

where C_2 is a second constant of integration.

Boundary conditions:

At the free surface, the shear stress is zero:

$$\tau_{xy} = \mu\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right)$$

or:

$$\frac{\partial u}{\partial y} = 0$$

This boundary condition gives:

$$C_1 = \frac{\rho g h\sin(\alpha)}{\mu}$$

At $y = 0$, the velocity along the plate is zero (no-slip condition), which gives:

$$C_2 = 0$$

Thus, the velocity profile is:

$$u(y) = \frac{\rho g\sin(\alpha)}{\mu}\left(yh - \frac{y^2}{2}\right)$$

3. The volume flow rate per unit width is:

$$q = \int_0^h u(y)\,dy$$

Substituting the expression for $u(y)$:

$$q = \frac{\rho g \sin(\alpha)}{\mu} \int_0^h \left(yh - \frac{y^2}{2} \right) dy$$

Evaluating the integral:

$$q = \frac{\rho g \sin(\alpha)}{\mu} \left[\frac{y^2 h}{2} - \frac{y^3}{6} \right]_0^h$$

Simplifying:

$$q = \frac{\rho g h^3 \sin(\alpha)}{3\mu}$$

Substitute the values:

$$q = \frac{1000\,\mathrm{kg/m^3} \cdot 9.81\,\mathrm{m/s^2} \cdot (0.001\,\mathrm{m})^3 \cdot \sin(15^\circ)}{3 \cdot 0.00152\,\mathrm{Ns/m^2}} = 0.000557\,\mathrm{m^2/s}$$