Fluid Mechanics 4 (Mechanical)
Tutorial week 5

Question 1

Consider a block of mass 30 kg that is being pulled across a flat plate at a constant velocity of 2 m/s, as shown in Fig. 1. The area of the block's bottom surface is 350 cm^2.

1. Determine the force required to pull the block when there is no oil, assuming the coefficient of kinetic friction is 0.2. This part of the problem is not a fluid problem, but rather a simple dynamics problem to benchmark how a lubricating oil can reduce the force required to move an object.

2. Determine the force required to pull the block if a thin layer of lubricating oil of height 0.25 mm exists between the block and flat plate. Assume the oil to be a Newtonian fluid with dynamic viscosity $\mu = 0.095\,\text{Ns/m}^2$. Assume that the pressure gradient within the travelling direction is negligible.

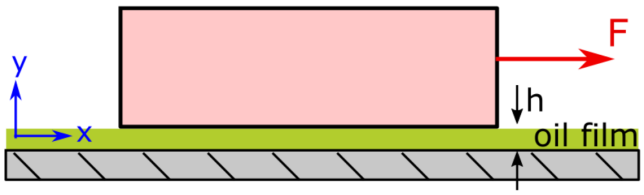


Figure 1: Oil film between a block and a surface.

Solution

1. The force needed to move a mass at constant velocity with kinetic friction coefficient is:

$$F = \mu_N mg = 0.2 \times 30\,\text{kg} \times 9.81\,\text{m/s}^2 = 58.9\,\text{N}$$

2. With a lubricating oil, it is the shear stress acting at the interface of the block and the lubricating oil that resists the block's motion. To calculate the shear stress $\mu\left(\frac{\partial u}{\partial y}\right)$, we should first find an expression of the velocity profile $u(y)$. This flow situation represents a Couette flow without a pressure gradient ($\partial p/\partial x = 0$).

   Assumptions:

   - The fluid flow is primarily in the x-direction (i.e., $v = w = 0$).
   - The flow is steady (constant velocity, $\partial u/\partial t = 0$).
   - $\partial p/\partial x = 0$, given
   - The dimensions of the block are much larger than the film thickness (changes in velocity in the z-direction are negligible, $\partial/\partial z = 0$)
   - Oil is incompressible

Write equations of motion

Continuity:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0$$

For incompressible flow:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

If v and w are negligible, the continuity equation reduces to:

$$\frac{\partial u}{\partial x} = 0$$

This implies that u is not changing in the x-direction.

x-momentum equation:

$$\rho \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \right) = \rho g_x - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + \mu \frac{1}{3} \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)$$

therefore

$$\rho \left(0 + u \cdot 0 + 0 \cdot \frac{\partial u}{\partial y} + 0 \cdot 0 \right) = 0 - 0 + \mu \left(0 + \frac{\partial^2 u}{\partial y^2} + 0 \right)$$

This reduces to:

$$\frac{\partial^2 u}{\partial y^2} = 0$$

Now, integrate with respect to y:

$$\frac{\partial u}{\partial y} = C_1$$

where C_1 is a constant of integration. Integrating again with respect to y:

$$u = C_1 y + C_2$$

where C_2 is another constant.

Applying boundary conditions:

At $y = 0$, the velocity of the fluid is zero (no-slip condition), which gives:

$$C_2 = 0$$

At $y = h$, the velocity is equal to the velocity of the block, U. This gives:

$$C_1 = \frac{U}{h}$$

Therefore, the velocity profile is:

$$u(y) = \frac{U}{h} y$$

This represents a simple Couette flow.

Shear stress at the interface:

$$\tau_{xy} = \mu \frac{\partial u}{\partial y} = \mu \frac{U}{h}$$

Shear force resisting motion:

$$F_{\text{shear}} = \tau_{xy} A = \mu \frac{U}{h} A$$

Substitute the values:

$$F_{\text{shear}} = 0.095\,\text{Ns/m}^2 \cdot 2\,\text{m/s} \cdot (0.00025\,\text{m})^{-1} \cdot 0.035\,\text{m}^2 = 26.6\,\text{N}$$

The force is less than half of what it would be without a lubricating oil.

Exam-style questions

Question 2

A new design for a wind turbine rotor is placed in a wind tunnel to determine the drag it will impose on the tower (see Fig. 2). The tests are performed with a uniform inlet velocity of 30 m/s. The pressure measured in section 1 and section 2 of the wind tunnel are measured to be atmospheric pressure (100 kPa). The velocity distribution at section 2, which is symmetrical about the centreline, is given by:

$$u = 30 - 5\left(1 - \frac{y}{3}\right) \text{ m/s} \quad |y| \leq 1\,\text{m}$$
$$u = 30\,\text{m/s} \quad |y| > 1\,\text{m}$$

Where y is the distance (in m) on either side of the centreline. Assume that the object's shape does not change in the in-plane direction (i.e., 2D) and that the flow is steady. Calculate the drag force (i.e., the reaction force in the x-direction) per unit width exerted on the air by the object. Assume the air to have ambient density $\rho = 1.2\,\text{kg/m}^3$.

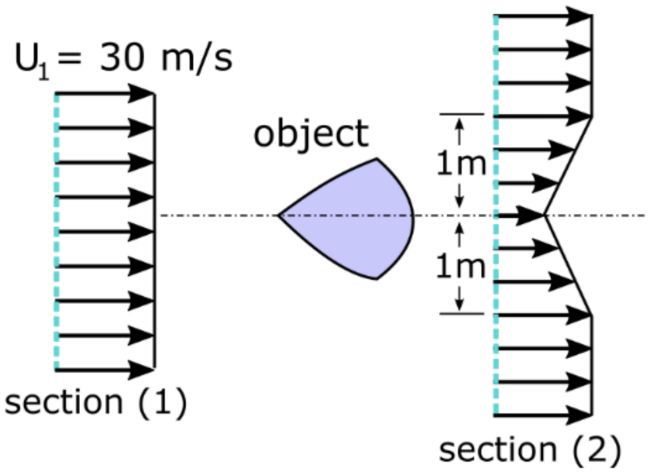


Figure 2: Object in a wind tunnel.

Solution

Choose a control volume along a streamline that contains the section where velocity differs from section 1 to section 2. Such a control volume will only have mass entering and leaving normal to sections 1 and 2.
Apply the integral form of the momentum equation.

$$\underbrace{\rho \int_S u_i u_j n_j dS}_{\text{net momentum flux leaving CS (out-in)}} = \underbrace{-\int_S p n_i dS}_{\text{Pressure forces acting on CS}} + \underbrace{\mu \int_S \left(\frac{\partial u_i}{\partial x_j}\right) n_j dS}_{\text{Viscous forces acting on CS}}$$

The RHS of the equation will be equal to the drag force $-F_{\text{drag}}$ acting in the negative x-direction.

$$\rho(-U_1)U_1 bh + 2\int_0^1 \rho u_2 u_2 b dy = -F_{\text{drag}}$$

Where b is the direction into the plane and h is the height of the inlet section. Inserting the expression for u_2 gives:

$$\frac{F_{\text{drag}}}{b} = \rho U_1^2 h - 2\rho \int_0^1 \left[30 - 5\left(1 - \frac{y}{3}\right)\right]^2 dy$$

To solve the second term on the RHS, one needs to apply a $u\,du$ substitution.

Let:

$$u = 30 - 5\left(1 - \frac{y}{3}\right), \quad du = \frac{5}{3}dy$$

Thus:

$$= 2\rho\frac{3}{5}\int u^2 du = 2\rho\frac{3}{5}\frac{u^3}{3} = \frac{2}{5}\rho u^3$$

Evaluate the integral:

$$\frac{2}{5}\rho\left[30 - 5\left(1 - \frac{y}{3}\right)\right]^3\Bigg|_0^1 = \frac{2}{5}\rho\left[\left(30 - 5\left(1 - \frac{1}{3}\right)\right)^3 - \left(30 - 5\left(1 - \frac{0}{3}\right)\right)^3\right]$$

$$= 1335 \cdot \rho = 1602\,\mathrm{N/m}$$

The challenge now is to determine h in the first term on the RHS of the momentum equation. Remember, h is the height of the inlet velocity section that encompasses the streamline section of interest. To do this, consider the conservation of mass:

$$\frac{DM_{\mathrm{sys}}}{Dt} = \frac{\partial}{\partial t}\int_{CV}\rho dV + \int_{CS}\rho\vec{V}\cdot\vec{n}dA = 0$$

If the flow is steady, $\frac{\partial}{\partial t} = 0$. Since the control volume considers streamlines to be the top and bottom boundary, there is no flow into or out of the control volume from the top or bottom surfaces. The only flow entering or leaving the control volume is from sections 1 and 2.

$$\int_{CS}\rho\vec{V}\cdot\vec{n}dA = 0 = \rho(-U_1)bh + \rho 2\int_0^1 u_2 b dy$$

$$\rho U_1 h = \rho 2\int_0^1\left(30 - 5\left(1 - \frac{y}{3}\right)\right)dy = \rho 2\left[30y - 5\left(y - \frac{y^2}{6}\right)\right]_0^1 = 51.7\cdot\rho$$

Solving for h gives:

$$\rho U_1 h = 51.7\cdot\rho \quad \Rightarrow \quad h = \frac{51.7}{U_1} = \frac{51.7\frac{m^2}{s}}{30\frac{m}{s}} = 1.72\,\mathrm{m}$$

Substitute this information back into the x-momentum equation to solve for F_{drag}:

$$\frac{F_{\mathrm{drag}}}{b} = \rho U_1^2 h - 2\rho\int_0^1\left(30 - 5\left(1 - \frac{y}{3}\right)\right)^2 dy$$

$$\frac{F_{\mathrm{drag}}}{b} = \rho U_1^2 h - 1335\cdot\rho$$

$$F_{\mathrm{drag}} = 1.2\frac{kg}{m^3}\cdot\left(30\frac{m}{s}\right)^2\cdot 1.72\,\mathrm{m} - 1335\frac{m^3}{s^2}\cdot 1.2\frac{kg}{m^3} = 255.6\frac{N}{m}.$$

This force acts in the negative x-direction.