# Fluid Mechanics 4 (Mechanical)
# Tutorial week 4

## Question 1

Consider a fluid flowing through a cylindrical pipe of varying diameter. The velocity field of the fluid is given by:

$$\mathbf{u}(x, r) = (v(x, r), 0, 0)$$

where $v(x, r)$ is the axial velocity, which varies with the axial $x$ and radial $r$ positions. The mass flow rate $\dot{m}$ at any cross-section of the pipe can be expressed as:

$$\dot{m} = \int_A \rho \mathbf{u} \cdot \mathbf{n} \, dA$$

where $\rho$ is the fluid density and $A$ is the cross-sectional area.

- Derive the expression for mass conservation for an incompressible fluid with constant density in a control volume that includes a segment of the pipe with varying diameter.
- If the diameter of the pipe at section 1 is $D_1$ and at section 2 is $D_2$, and the fluid is incompressible with constant density $\rho$, express the relationship between the velocities $v_1$ and $v_2$ at these two sections, assuming that the velocities $v_1$ and $v_2$ are spatially homogeneous over the sections (independent of $r$).

### Solution

**Derive the Expression for Mass Conservation**
The conservation of mass states that the rate of mass entering a control volume must equal the rate of mass exiting the control volume, plus the rate of change of mass within the control volume $V_0$. For a control volume with fluid flowing in and out, the mass conservation equation can be written as:

$$\int_{V_0} \frac{\partial \rho}{\partial t} \, dV + \int_{S_0} \rho \mathbf{u} \cdot \mathbf{n} \, dS = 0$$

For a steady flow of an incompressible fluid with constant density $\rho$, the equation simplifies to:

$$-\int_{A_{in}} v \, dA + \int_{A_{out}} v \, dA = 0$$

or

$$\int_{A_{in}} v \, dA = \int_{A_{out}} v \, dA$$

This is the integral form of the continuity equation for incompressible flow in a section of a pipe with varying cross-sectional area.
**Express the Relationship Between the Velocities $v_1$ and $v_2$**
Let's denote the area of the cross-section at section 1 as $A_1$ and at section 2 as $A_2$. For a circular pipe, the area can be expressed as:

$$A = \frac{\pi D^2}{4}$$

Therefore, the areas at the two sections are:

$$A_1 = \frac{\pi D_1^2}{4}, \quad A_2 = \frac{\pi D_2^2}{4}$$

From the continuity equation, we have:

$$A_1 v_1 = A_2 v_2$$

Substituting the expressions for the areas gives:

$$\frac{\pi D_1^2}{4} v_1 = \frac{\pi D_2^2}{4} v_2$$

Canceling $\frac{\pi}{4}$ from both sides results in:

$$D_1^2 v_1 = D_2^2 v_2$$

Rearranging gives the relationship between the velocities at the two sections:

$$v_2 = v_1 \left( \frac{D_1^2}{D_2^2} \right)$$

# Question 2

A rigid, well-mixed tank of volume $V_{\text{tank}} = 2$ m$^3$ contains a gas initially at uniform density $\rho_0 = 1$ kg/m$^3$. A feed line delivers gas at a constant *mass* flow rate $\dot{m}_{\text{in}} = 0.2$ kg/s. Gas exits through a valve whose *volumetric* outflow rate is constant, $\dot{V}_{\text{out}} = 0.1$ m$^3$/s. Because the tank is perfectly mixed, the density of the outflow equals the instantaneous tank density, $\rho(t)$. Assume single phase, no leaks, and spatially uniform properties in the tank.

1. Write the **Reynolds Transport Theorem (RTT)** for mass and derive the ODE (Ordinary Differential Equation) for the tank mass $m(t)$.
2. Solve the equation derived in the previous point to obtain $m(t)$ and $\rho(t)$ given $m(0) = \rho_0 V_{\text{tank}}$. The equation derived in the previous point is not easy to solve. If you do not remember how to solve it, the solution is:

   $$m(t) = m_0 \, e^{-kt} + \frac{\dot{m}_{\text{in}}}{k} \Big( 1 - e^{-kt} \Big).$$

   with $k \equiv \dot{V}_{\text{out}} / V_{\text{tank}}$.

   While the math here is more complex than typical exam questions, this exercise is useful to understand the link between the conservation equation and a real problem.
3. If the gas is ideal at constant $T = 300$ K with $R = 287$ J/(kg K), give $p(t)$.

## Solution

**1) RTT for mass & tank balance.** For the extensive property $B =$ mass (so $b = 1$), the RTT reads

$$\int_{V_0} \frac{\partial \rho}{\partial t} \, dV + \int_{S_0} \rho \mathbf{u} \cdot \mathbf{n} \, dS = 0$$

so

$$\frac{d}{dt} \int_{V_0} \rho \, dV = \dot{m}_{\text{in}} - \dot{m}_{\text{out}}.$$

For a rigid tank $V_{\text{tank}} = \text{const}$, the total mass in the tank is

$$m(t) = \int_{V_0} \rho \, dV = \rho(t) V_{\text{tank}}.$$

With perfect mixing, the mass flow out equals the product of density in the tank and the volumetric outflow:

$$\dot{m}_{\text{out}} = \rho(t)\,\dot{V}_{\text{out}}.$$

Hence the unsteady mass balance becomes

$$\frac{dm}{dt} \;=\; \dot{m}_{\text{in}} \;-\; \rho(t)\,\dot{V}_{\text{out}} \;=\; \dot{m}_{\text{in}} \;-\; \frac{\dot{V}_{\text{out}}}{V_{\text{tank}}}\,m(t).$$

Define $k \equiv \dot{V}_{\text{out}}/V_{\text{tank}}$. With the given data, $k = 0.1/2 = 0.05\ \text{s}^{-1}$. Thus

$$\boxed{\frac{dm}{dt} + k\,m = \dot{m}_{\text{in}}.}$$

**2) Solve for $m(t)$ and $\rho(t)$** Starting from the ODE derived in Part 1,

$$\frac{dm}{dt} + k\,m = \dot{m}_{\text{in}}, \qquad k \equiv \frac{\dot{V}_{\text{out}}}{V_{\text{tank}}}\ (>0), \qquad m(0) = m_0 = \rho_0 V_{\text{tank}}.$$

This is a linear, first-order, nonhomogeneous ODE. Multiply the ODE by $e^{kt}$:

$$e^{kt}\frac{dm}{dt} + ke^{kt}m = \dot{m}_{\text{in}}\,e^{kt} \;\;\Longrightarrow\;\; \frac{d}{dt}\big(m\,e^{kt}\big) = \dot{m}_{\text{in}}\,e^{kt}.$$

Integrate from 0 to $t$:

$$m(t)\,e^{kt} - m_0 = \dot{m}_{\text{in}}\int_0^t e^{ks}\,ds = \dot{m}_{\text{in}}\,\frac{e^{kt}-1}{k}.$$

Solve for $m(t)$:

$$m(t) = m_0\,e^{-kt} + \frac{\dot{m}_{\text{in}}}{k}\Big(1 - e^{-kt}\Big).$$

It is often clearer to write this using the steady value

$$m_\infty \;\equiv\; \frac{\dot{m}_{\text{in}}}{k} = \frac{\dot{m}_{\text{in}}}{\dot{V}_{\text{out}}/V_{\text{tank}}} = \frac{V_{\text{tank}}\,\dot{m}_{\text{in}}}{\dot{V}_{\text{out}}},$$

so that

$$\boxed{m(t) = m_\infty + \big(m_0 - m_\infty\big)\,e^{-kt}}.$$

Define the time constant $\tau \equiv 1/k = V_{\text{tank}}/\dot{V}_{\text{out}}$, so $e^{-kt} = e^{-t/\tau}$. With the given numbers,

$$k = \frac{0.1}{2} = 0.05\ \text{s}^{-1}, \qquad \tau = \frac{1}{k} = 20\ \text{s}, \qquad m_0 = \rho_0 V_{\text{tank}} = 1 \times 2 = 2\ \text{kg},$$

$$m_\infty = \frac{V_{\text{tank}}\,\dot{m}_{\text{in}}}{\dot{V}_{\text{out}}} = \frac{2 \times 0.2}{0.1} = 4\ \text{kg}.$$

Hence

$$\boxed{m(t) = 4 + \big(2-4\big)e^{-t/20} = 4 - 2\,e^{-0.05\,t}\ \text{kg}.}$$

Since $\rho(t) = m(t)/V_{\text{tank}}$,

$$\boxed{\rho(t) = \frac{m(t)}{V_{\text{tank}}} = \frac{4 - 2\,e^{-0.05t}}{2} = 2 - e^{-0.05t}\ \text{kg/m}^3.}$$

**Sanity checks and interpretation.**

- **Initial value:** $m(0) = m_\infty + (m_0 - m_\infty) = m_0$ (satisfies the IC).
- **Long time:** $t \to \infty \Rightarrow e^{-kt} \to 0$, so $m(t) \to m_\infty$, i.e. the tank relaxes to a steady mass where inflow equals outflow: $\dot{m}_{\text{in}} = \rho_\infty \dot{V}_{\text{out}}$.
- **Steady density:** $\rho_\infty = m_\infty/V_{\text{tank}} = \dot{m}_{\text{in}}/\dot{V}_{\text{out}} = 2\ \text{kg/m}^3$.
- **Monotonic approach:** Because $m_0 < m_\infty$, $m(t)$ increases monotonically to $m_\infty$ with time constant $\tau = 20$ s.
- **Dimensional check:** $k = \dot{V}_{\text{out}}/V_{\text{tank}}$ has units 1/s; thus $e^{-kt}$ is dimensionless and $m(t)$ has units of kg as required.

**3) Ideal-gas pressure.** With constant $T$ and ideal gas,

$$p(t) = \rho(t)\, RT = \left[2 - e^{-0.05\,t}\right] (287)\,(300)\ \text{Pa}.$$

Since $RT = 86100\ \text{Pa}\,\text{m}^3/\text{kg}$,

$$\boxed{p(t) = 86100\left[2 - e^{-0.05\,t}\right]\ \text{Pa}.}$$

# Question 3

Consider a fluid with density $\rho(\mathbf{x}, t)$ and velocity field $\mathbf{u}(\mathbf{x}, t)$.

(a) Starting from the **integral mass balance** over a fixed control volume $V_0$ with control surface $S_0$,

$$\int_{V_0} \frac{\partial \rho}{\partial t}\, dV + \int_{S_0} \rho\, \mathbf{u} \cdot \mathbf{n}\, dS = 0,$$

use the divergence theorem to derive the **differential form** of mass conservation. State the mathematical conditions under which your steps are valid.

(b) Expand your result to obtain $\dfrac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho + \rho\, \nabla \cdot \mathbf{u} = 0$, and rewrite it using the material derivative $D/Dt$. Briefly interpret each term.

(c) If $\rho = \rho_0 = \text{const}$ in space and time, show that $\nabla \cdot \mathbf{u} = 0$.

(d) Check incompressibility for the follwoing flows.

(i) $\mathbf{u}(x, y, z) = (ax,\, by,\, cz)$. Find the condition on $a, b, c$ for incompressibility.

(ii) $\mathbf{u}(x, y) = (\alpha x,\, -\alpha y)$ (planar stagnation flow). Is it incompressible?

(iii) $\mathbf{u}(x, y) = \left(-\dfrac{\Gamma}{2\pi}\dfrac{y}{r^2},\, \dfrac{\Gamma}{2\pi}\dfrac{x}{r^2}\right)$, where $r^2 = x^2 + y^2$ (potential vortex). Evaluate $\nabla \cdot \mathbf{u}$ for $r \neq 0$.

## Solution

**(a) From integral to differential form.** Apply the divergence theorem to the surface integral:

$$\int_{S_0} \rho\, \mathbf{u} \cdot \mathbf{n}\, dS = \int_{V_0} \nabla \cdot (\rho \mathbf{u})\, dV.$$

Hence,

$$\int_{V_0} \left( \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \right) dV = 0.$$

Because this holds for any fixed and arbitrary control volume $V_0$, the integrand must vanish:

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0.}$$

*Conditions:* $\rho$ and $\mathbf{u}$ must be continuous with piecewise continuous derivatives, so that the divergence theorem applies.

**(b) Convective form and material derivative.** Expand the divergence term:

$$\nabla \cdot (\rho \mathbf{u}) = u_x \frac{\partial \rho}{\partial x} + u_y \frac{\partial \rho}{\partial y} + u_z \frac{\partial \rho}{\partial z} + \rho \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right).$$

Thus,

$$\frac{\partial \rho}{\partial t} + u_x \frac{\partial \rho}{\partial x} + u_y \frac{\partial \rho}{\partial y} + u_z \frac{\partial \rho}{\partial z} + \rho \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right) = 0.$$

Introducing the material derivative,

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + u_x \frac{\partial \rho}{\partial x} + u_y \frac{\partial \rho}{\partial y} + u_z \frac{\partial \rho}{\partial z},$$

we obtain the compact form:

$$\boxed{\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0.}$$

*Interpretation:* - $D\rho/Dt$ is the rate of density change experienced by a moving fluid particle. - $\rho \nabla \cdot \mathbf{u}$ represents the rate of volumetric expansion (+) or compression (−).

**(c) Incompressible limit.** For a constant density $\rho = \rho_0$, the material derivative vanishes ($D\rho/Dt = 0$), giving:

$$\boxed{\nabla \cdot \mathbf{u} = 0.}$$

This is the incompressibility condition.

**(d) Divergence checks.**

(i) For $\mathbf{u}(x, y, z) = (ax,\ by,\ cz)$,

$$\nabla \cdot \mathbf{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = a + b + c.$$

Incompressible if and only if $\boxed{a + b + c = 0.}$

(ii) For $\mathbf{u}(x, y) = (\alpha x,\ -\alpha y)$,

$$\nabla \cdot \mathbf{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = \alpha - \alpha = 0,$$

so the flow is $\boxed{\text{incompressible.}}$

(iii) For

$$\mathbf{u}(x, y) = \left( -\frac{\Gamma}{2\pi} \frac{y}{r^2},\ \frac{\Gamma}{2\pi} \frac{x}{r^2} \right), \quad r^2 = x^2 + y^2,$$

we find

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0 \quad \text{for } r \neq 0.$$

Hence the potential vortex is $\boxed{\text{incompressible for } r > 0.}$

# Exam-style questions

# Question 4

The density in a horizontal flow $u = U(y, z)\, \boldsymbol{e}_x$ is given by $\rho(x, y, z) = f(x - Ut, y, z)$, where $f(x, y, z)$ is the density distribution at $t = 0$. Is this flow incompressible?

## Solution

$$\nabla \cdot \mathbf{u} = \frac{\partial U(y, z)}{\partial x} + 0 + 0 = 0.$$

so the flow is incompressible. Note that the density is not constant in this flow if $f$ is not constant. However, incompressible means that $\nabla \cdot \mathbf{u} = 0$, which is verified here.