Fluid Mechanics 4 (Mechanical)
Tutorial week 3

Question 1

Consider the steady two-dimensional velocity field

$$\mathbf{u}(x, y) = (u, v) = (2x,\, -2y).$$

1. Derive the equation of the streamlines.
2. Sketch the family of streamlines in the xy-plane.

Solution

Given $\mathbf{u}(x, y) = (u, v) = (2x, -2y)$.

1) Streamlines. By definition,

$$\frac{dy}{dx} = \frac{v}{u} = \frac{-2y}{2x} = -\frac{y}{x}.$$

Starting from

$$\frac{dy}{dx} = -\frac{y}{x} \qquad \text{(we assume } u \neq 0 \text{ so } x \neq 0 \text{ in this derivation)},$$

separate variables:

$$\frac{1}{y}\, dy = -\frac{1}{x}\, dx.$$

Integrate both sides.

$$\int \frac{1}{y}\, dy = \int -\frac{1}{x}\, dx \quad \Longrightarrow \quad \ln|y| = -\ln|x| + C,$$

where C is an arbitrary constant.
Combine the logarithms.

$$\ln|y| + \ln|x| = C \quad \Longrightarrow \quad \ln\big(|x|\,|y|\big) = \ln|xy| = C.$$

Exponentiate to remove the log.

$$|xy| = e^{C} = C_1, \qquad C_1 > 0.$$

Absorb the absolute value into the constant. Since $|xy| = C_1$ with $C_1 > 0$ can represent both $xy = C_1$ and $xy = -C_1$, we can write the family with a single real constant $C \in R$ (including $C = 0$):

$$\boxed{xy = C}.$$

This captures streamlines in quadrants I and III when $C > 0$, and in II and IV when $C < 0$. The case $C = 0$ corresponds to the separatrices $x = 0$ or $y = 0$.

Special cases (consistency check). The streamline ODE $\frac{dy}{dx} = v/u$ assumes $u \neq 0$ (i.e. $x \neq 0$). The line $x = 0$ is nonetheless a valid streamline of the vector field itself (there $u = 0$ and $v = -2y$ gives purely vertical motion). Likewise, $y = 0$ is a streamline (purely horizontal motion). These are exactly the $C = 0$ members of $xy = C$ in the limiting sense.

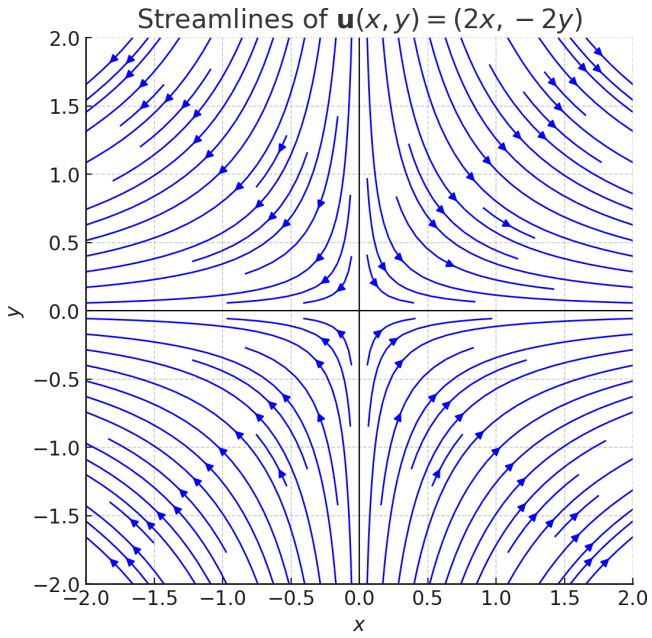


Figure 1: Streamlines

2) Sketch. The streamlines $xy = C$ are hyperbolae with asymptotes along the coordinate axes. Curves with $C > 0$ lie in quadrants I and III; with $C < 0$ in quadrants II and IV (see Fig. 1).

Question 2

Consider the two-dimensional velocity field

$$\mathbf{u}(x, y) = (u, v) = \big(1.5,\ -0.6\, y\big),$$

with SI units (m, s). A fluid particle starts at $(x(0), y(0)) = (x_0, y_0)$.

1. Write the ODEs for the particle trajectory $(x(t), y(t))$.
2. Solve for $x(t)$ and $y(t)$ for arbitrary x_0, y_0.
3. Eliminate t to obtain the particle path in the (x, y)–plane.

Solution

1) Trajectory ODEs.

$$\frac{dx}{dt} = u = 1.5, \qquad \frac{dy}{dt} = v = -0.6\, y.$$

2) Solve with $(x(0), y(0)) = (x_0, y_0)$. The x–equation integrates directly:

$$x(t) = x_0 + 1.5\, t.$$

The y–equation is linear homogeneous:

$$\frac{dy}{dt} = -0.6\, y \ \Rightarrow\ \frac{1}{y}\, dy = -0.6\, dt \ \Rightarrow\ \ln|y| = -0.6\, t + C \ \Rightarrow\ y(t) = y_0\, e^{-0.6\, t}.$$

3) Eliminate t to get the path in (x, y). From $x(t) = x_0 + 1.5\,t$ we have $t = \dfrac{x - x_0}{1.5}$. Substitute into $y(t)$:

$$y(x) = y_0 \, \exp\Big(-0.6 \, \frac{x - x_0}{1.5} \Big) = y_0 \, e^{-0.4(x - x_0)} = \big(y_0 \, e^{0.4 x_0} \big) \, e^{-0.4x}.$$

Thus particle paths are exponential curves in x:

$$\boxed{y(x) = C \, e^{-0.4x}, \quad C = y_0 \, e^{0.4 x_0}}.$$

Question 3

Consider a horizontal pipe of cross-sectional area $A = 0.05$ m^2 carrying a **gas** with density $\rho = 1$ kg/m^3 at the time considered here (ρ is not necessarily constant in time). The velocity field is uniform across the section.
At the pipe inlet, the velocity is $u_{\text{in}} = 2$ m/s. At the pipe outlet, the velocity is $u_{\text{out}} = 3$ m/s.

1. Using the **Reynolds Transport Theorem** with $b = 1$ (so B is the total mass), write the mass conservation equation for a fixed control volume enclosing the pipe segment.
2. Compute the net rate of mass flow out of the control volume (in kg/s).
3. State whether the mass inside the control volume increases, decreases, or remains constant.

Solution

Given: gas with $\rho = 1$ kg/m^3, area $A = 0.05$ m^2, $u_{\text{in}} = 2$ m/s, $u_{\text{out}} = 3$ m/s.

1) RTT for mass (with $b = 1$). For a fixed control volume V_0 bounded by the surface S_0,

$$0 \;=\; \frac{d}{dt} \int_{V_0} \rho \, dV \;+\; \oint_{S_0} \rho \, \mathbf{u} \cdot \mathbf{n} \, dS,$$

i.e. $0 = \dfrac{dM_{V_0}}{dt} + \displaystyle\sum_{\text{faces}} \dot{m}_{\text{out}} - \dot{m}_{\text{in}}$. Here $\mathbf{n}$ is the outward unit normal on the control surface.

2) Mass flow rates at inlet and outlet. Because the velocity and density are uniform on each section,

$$\dot{m}_{\text{in}} = \rho \, u_{\text{in}} \, A = (1)(2)(0.05) = \boxed{0.10 \text{ kg/s}},$$

$$\dot{m}_{\text{out}} = \rho \, u_{\text{out}} \, A = (1)(3)(0.05) = \boxed{0.15 \text{ kg/s}}.$$

3) Net rate of mass flow out of the control volume.

$$\dot{m}_{\text{net,out}} = \dot{m}_{\text{out}} - \dot{m}_{\text{in}} = 0.15 - 0.10 = \boxed{0.05 \text{ kg/s}}.$$

4) Accumulation inside the control volume. From RTT:

$$0 = \frac{dM_{V_0}}{dt} + \dot{m}_{\text{net,out}} \quad \Longrightarrow \quad \frac{dM_{V_0}}{dt} = -\,\dot{m}_{\text{net,out}} = \boxed{-0.05 \text{ kg/s}}.$$

Interpretation: since outflow > inflow, mass inside the control volume *decreases* at 0.05 kg/s.

5) Sign convention note. At the outlet, $\mathbf{u} \cdot \mathbf{n} > 0$ (flow leaves the control volume); at the inlet, $\mathbf{u} \cdot \mathbf{n} < 0$ (flow enters), which is why the surface integral naturally gives "outflow minus inflow" in the RTT.

Exam-style questions

Question 4

For a fixed control volume, the Reynolds Transport Theorem can be written as

$$\frac{dB}{dt} = \int_{V_0} \frac{\partial(\rho b)}{\partial t}\, dV + \int_{S_0} \rho b\, \mathbf{u} \cdot \mathbf{n}\, dS.$$

Briefly explain the physical meaning of the two terms on the right-hand side.

Solution

The first term,

$$\int_{V_0} \frac{\partial(\rho b)}{\partial t}\, dV,$$

represents the rate of accumulation (or depletion) of the quantity B inside the control volume.
The second term,

$$\int_{S_0} \rho b\, \mathbf{u} \cdot \mathbf{n}\, dS,$$

represents the net rate at which the quantity B is transported out of the control volume through its surface.
Therefore, the Reynolds Transport Theorem relates the rate of change of a quantity for a material volume to its accumulation inside a control volume and its net flux through the control surface.