Fluid Mechanics 4 (Mechanical)
Tutorial week 2

Question 1

An anemometer is located at a point:

$$\vec{P}_1 = (0, 0) \text{ m},$$

near an airplane wing and measured a constant velocity of:

$$\vec{u} = (10, 10) \text{ m/s} \quad \text{(i.e., 2D flow)}$$

When the anemometer was moved 0.1 m in the x-direction to point:

$$\vec{P}_2 = (0.1, 0) \text{ m},$$

it measured a velocity of:

$$\vec{u} = (10.1, 10.1) \text{ m/s}.$$

When the anemometer was moved 0.1 m in the y-direction to point:

$$\vec{P}_3 = (0.0, 0.1) \text{ m},$$

it measured a velocity of:

$$\vec{u} = (10.2, 10.2) \text{ m/s}.$$

Estimate:
(a) the local acceleration,
(b) the convective acceleration, and
(c) the total acceleration.

Solution

Given Data:

$$\vec{P}_1 = (0, 0)\,\text{m}, \quad \vec{u_1} = (10, 10)\,\text{m/s}$$
$$\vec{P}_2 = (0.1, 0)\,\text{m}, \quad \vec{u_2} = (10.1, 10.1)\,\text{m/s}$$
$$\vec{P}_3 = (0, 0.1)\,\text{m}, \quad \vec{u_3} = (10.2, 10.2)\,\text{m/s}$$

1. Local Acceleration ($\frac{\partial \vec{u}}{\partial t}$):

$$\frac{\partial \vec{u}}{\partial t} = 0 \quad \text{(since the flow is steady)}$$

2. Convective Acceleration:

$$\vec{u} \cdot \nabla \vec{u}$$

We first calculate the partial derivatives:

$$\frac{\partial u_x}{\partial x} = \frac{u_{2x} - u_{1x}}{x_2 - x_1} = \frac{10.1 - 10}{0.1 - 0} = 1\,\text{m/s}^2$$

$$\frac{\partial u_y}{\partial x} = \frac{u_{2y} - u_{1y}}{x_2 - x_1} = \frac{10.1 - 10}{0.1 - 0} = 1\,\mathrm{m/s^2}$$
$$\frac{\partial u_x}{\partial y} = \frac{u_{3x} - u_{1x}}{y_3 - y_1} = \frac{10.2 - 10}{0.1 - 0} = 2\,\mathrm{m/s^2}$$
$$\frac{\partial u_y}{\partial y} = \frac{u_{3y} - u_{1y}}{y_3 - y_1} = \frac{10.2 - 10}{0.1 - 0} = 2\,\mathrm{m/s^2}$$

Using:

$$a_x = u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y}$$

$$a_y = u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y}$$

Thus, the convective acceleration is:

$$\vec{u} \cdot \nabla \vec{u} = (30, 30)\,\mathrm{m/s^2}$$

3. Total Acceleration:

$$\vec{a} = \frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u}$$

Since the local acceleration is zero, the total acceleration is:

$$\vec{a} = (30, 30)\,\mathrm{m/s^2}$$

Question 2

Consider the two-dimensional velocity field

$$\mathbf{u}(x, y) = (\gamma y,\ 0),$$

where γ is a constant.

1. Compute the velocity gradient tensor $\nabla \mathbf{u}$.
2. Decompose $\nabla \mathbf{u}$ into:
   - the volumetric dilation (compression/expansion),
   - the symmetric strain rate tensor E,
   - the antisymmetric rotation tensor Ω.
3. Determine the vorticity vector $\boldsymbol{\omega} = \nabla \times \mathbf{u}$.
4. Briefly describe the type of fluid deformation taking place.

Solution

Let $\mathbf{u}(x, y) = (u, v) = (\gamma y,\, 0)$ with constant γ.

1) Velocity gradient.

$$\nabla \mathbf{u} = \begin{bmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{bmatrix} = \begin{bmatrix} 0 & \gamma \\ 0 & 0 \end{bmatrix}.$$

2) Decomposition $\nabla \mathbf{u} = \frac{1}{3}(\nabla \cdot \mathbf{u})\,\mathbf{I} + E + \Omega$.

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \Rightarrow \quad A = \nabla \mathbf{u},\ E = \frac{A + A^{\mathsf{T}}}{2} = \begin{bmatrix} 0 & \frac{\gamma}{2} \\ \frac{\gamma}{2} & 0 \end{bmatrix},\ \Omega = \frac{A - A^{\mathsf{T}}}{2} = \begin{bmatrix} 0 & \frac{\gamma}{2} \\ -\frac{\gamma}{2} & 0 \end{bmatrix}.$$

(Decomposition into trace, symmetric E, and skew Ω.)

3) Vorticity.

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} = \big(0,\, 0,\, \partial v/\partial x - \partial u/\partial y\big) = (0,\, 0,\, -\gamma).$$

(With $\Omega \cdot \mathbf{x} = \frac{1}{2}\,\boldsymbol{\omega} \times \mathbf{x}$ and $\boldsymbol{\omega} = \nabla \times \mathbf{u}$.)

4) Physical description. Since $\nabla \cdot \mathbf{u} = 0$: no volumetric dilation. Nonzero $E_{xy} = E_{yx} = \gamma/2$: *angular (shear) deformation* at rate $\gamma/2$. Nonzero Ω: *rigid-body rotation* with spin $\widehat{\boldsymbol{\omega}} = \frac{1}{2}\boldsymbol{\omega} = (0, 0, -\gamma/2)$. This is a classic *simple shear* flow—combining shear deformation and a counter-rotating rigid-body component.

Question 3

Consider the scalar field $\phi(x, y, z) = x^2 + y^2 + z^2$ and the vector field $\mathbf{F}(\mathbf{x}) = (2xy, x^2 - z, 3z^2)$, where $\mathbf{x} = (x, y, z)$.

1. Calculate the gradient of the scalar field ϕ.
2. Calculate the divergence of the vector field $\mathbf{F}$.
3. Calculate the curl of the vector field $\mathbf{F}$.
4. Calculate the gradient of the vector field $\mathbf{F}$.
5. For all the quantities computed above, say if they are scalars, vectors, or second order tensors.

Solution

The gradient of a scalar field ϕ is given by:

$$\nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

For $\phi(x, y, z) = x^2 + y^2 + z^2$:

$$\frac{\partial\phi}{\partial x} = 2x, \quad \frac{\partial\phi}{\partial y} = 2y, \quad \frac{\partial\phi}{\partial z} = 2z$$

Thus, the gradient is:

$$\nabla\phi = (2x, 2y, 2z)$$

The divergence of a vector field $\mathbf{F} = (F_1, F_2, F_3)$ is given by:

$$\nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

For $\mathbf{F} = (2xy, x^2 - z, 3z^2)$:

$$\frac{\partial F_1}{\partial x} = \frac{\partial(2xy)}{\partial x} = 2y$$

$$\frac{\partial F_2}{\partial y} = \frac{\partial(x^2 - z)}{\partial y} = 0$$

$$\frac{\partial F_3}{\partial z} = \frac{\partial(3z^2)}{\partial z} = 6z$$

Therefore, the divergence is:

$$\nabla \cdot \mathbf{F} = 2y + 0 + 6z = 2y + 6z$$

The curl of a vector field $\mathbf{F} = (F_1, F_2, F_3)$ is given by:

$$\nabla \times \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

Calculating each component:

$$\frac{\partial F_3}{\partial y} = \frac{\partial (3z^2)}{\partial y} = 0, \quad \frac{\partial F_2}{\partial z} = \frac{\partial (x^2 - z)}{\partial z} = -1$$

$$\frac{\partial F_1}{\partial z} = \frac{\partial (2xy)}{\partial z} = 0, \quad \frac{\partial F_3}{\partial x} = \frac{\partial (3z^2)}{\partial x} = 0$$

$$\frac{\partial F_2}{\partial x} = \frac{\partial (x^2 - z)}{\partial x} = 2x, \quad \frac{\partial F_1}{\partial y} = \frac{\partial (2xy)}{\partial y} = 2x$$

Thus, the curl is:

$$\nabla \times \mathbf{F} = (0 - (-1), 0 - 0, 2x - 2x) = (1, 0, 0)$$

The gradient of a vector field $\mathbf{F} = (F_1, F_2, F_3)$ is given by

$$\nabla \mathbf{F} = \begin{bmatrix} \frac{\partial (2xy)}{\partial x} & \frac{\partial (2xy)}{\partial y} & \frac{\partial (2xy)}{\partial z} \\ \frac{\partial (x^2 - z)}{\partial x} & \frac{\partial (x^2 - z)}{\partial y} & \frac{\partial (x^2 - z)}{\partial z} \\ \frac{\partial (3z^2)}{\partial x} & \frac{\partial (3z^2)}{\partial y} & \frac{\partial (3z^2)}{\partial z} \end{bmatrix} = \begin{bmatrix} 2y & 2x & 0 \\ 2x & 0 & -1 \\ 0 & 0 & 6z \end{bmatrix}.$$

Question 4

Consider a fluid with the following velocity field in three-dimensional space:

$$\mathbf{u}(x, y, z, t) = (yz, x^2 - z, 2xy)$$

where $\mathbf{u} = (u_1, u_2, u_3)$ represents the velocity components in the x, y, and z directions, respectively. Calculate the components of the rate of strain tensor $\mathbf{S}$ for the given velocity field.

Solution

The rate of strain tensor $\mathbf{S}$ is defined as:

$$S_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

For the given velocity field $\mathbf{u} = (u_1, u_2, u_3) = (yz, x^2 - z, 2xy)$, we need to compute the derivatives:

$$\frac{\partial u_1}{\partial x} = 0, \quad \frac{\partial u_1}{\partial y} = z, \quad \frac{\partial u_1}{\partial z} = y$$

$$\frac{\partial u_2}{\partial x} = 2x, \quad \frac{\partial u_2}{\partial y} = 0, \quad \frac{\partial u_2}{\partial z} = -1$$

$$\frac{\partial u_3}{\partial x} = 2y, \quad \frac{\partial u_3}{\partial y} = 2x, \quad \frac{\partial u_3}{\partial z} = 0$$

Now we can compute the components of the rate of strain tensor $\mathbf{S}$:

$$S_{11} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x} + \frac{\partial u_1}{\partial x}\right) = 0$$
$$S_{22} = \frac{1}{2}\left(\frac{\partial u_2}{\partial y} + \frac{\partial u_2}{\partial y}\right) = 0$$
$$S_{33} = \frac{1}{2}\left(\frac{\partial u_3}{\partial z} + \frac{\partial u_3}{\partial z}\right) = 0$$
$$S_{12} = S_{21} = \frac{1}{2}\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right) = \frac{1}{2}(z + 2x) = \frac{z + 2x}{2}$$
$$S_{13} = S_{31} = \frac{1}{2}\left(\frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x}\right) = \frac{1}{2}(y + 2y) = \frac{3y}{2}$$
$$S_{23} = S_{32} = \frac{1}{2}\left(\frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial y}\right) = \frac{1}{2}(-1 + 2x) = \frac{2x - 1}{2}$$

Thus, the rate of strain tensor $\mathbf{S}$ can be expressed as:

$$\mathbf{S} = \begin{pmatrix} 0 & \frac{z+2x}{2} & \frac{3y}{2} \\ \frac{z+2x}{2} & 0 & \frac{2x-1}{2} \\ \frac{3y}{2} & \frac{2x-1}{2} & 0 \end{pmatrix}$$

Question 5

A fluid particle in a steady flow moves along the x-axis. Its distance from the origin is r_0 at time t_0, and its trajectory is $r(t) = \left[K(t - t_0) + r_0^3\right]^{1/3}$, where K is a positive constant with units of volume/time. Determine the flow Eulerian velocity and acceleration, $u(x)$ and $a(x)$, and show that $a(x)$ may also be obtained from $\frac{Du(x)}{Dt}$.

Solution

First, perform the differentiations to determine the particle Lagrangian velocity and acceleration:

$$u(t) = \frac{dr}{dt} = \frac{K}{3}\left[K(t - t_0) + r_0^3\right]^{-2/3}$$

and

$$a(t) = \frac{d^2r}{dt^2} = -\frac{2K^2}{9}\left[K(t - t_0) + r_0^3\right]^{-5/3}$$

To find the Eulerian fluid velocity and acceleration (functions of position), require $u(x)$ and $a(x)$ to be equal to $u(t)$ and $a(t)$ when $x = r(t)$. This step can be completed by substituting the given form of $r(t)$ into the equations for $u(t)$ and $a(t)$ to eliminate the particle-specification information (r_0 and t_0) as follows:

$$u(x) = \left[\frac{K}{3r^2}\right]_{x=r} = \frac{K}{3x^2}$$

and

$$a(x) = \left[-\frac{2K^2}{9r^5}\right]_{x=r} = -\frac{2K}{9x^5}$$

To show that this $a(x)$ may also be obtained from $\frac{Du(x)}{Dt}$, use the definition of $\frac{Du(x)}{Dt}$. Here, the Eulerian velocity does not depend on time and it is unidirectional, so the partial time derivative in $\frac{Du(x)}{Dt}$ drops out, and the dot product in $\frac{Du(x)}{Dt}$ simplifies to a single term. This lone dot-product term yields the desired result.

$$\frac{Du}{Dt} = \frac{\partial u(x)}{\partial t} + u(x)\frac{\partial u(x)}{\partial x} = 0 + \frac{K}{3x^2}\frac{\partial}{\partial x}\left(\frac{K}{3x^2}\right) = \frac{K}{3x^2}\left(-\frac{2K}{3x^3}\right) = -\frac{2K}{9x^5}$$

Question 6

Consider the two-dimensional velocity field

$$\mathbf{u}(x, y) = (ax,\ -ay),$$

where a is a constant.

1. Compute the velocity gradient tensor $\nabla\mathbf{u}$.
2. Decompose $\nabla\mathbf{u}$ into:
   - the volumetric dilation (compression/expansion),
   - the symmetric strain rate tensor E,
   - the antisymmetric rotation tensor Ω.
3. Determine the vorticity vector $\boldsymbol{\omega} = \nabla \times \mathbf{u}$.
4. Briefly describe the type of fluid deformation taking place.

Solution

Given $\mathbf{u}(x, y) = (ax,\ -ay)$ with constant a.

1) Velocity gradient. In 2D,

$$\nabla\mathbf{u} = \begin{bmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}.$$

2) Decomposition into volumetric part, E and Ω.

$$\nabla\mathbf{u} = \tfrac{1}{3}(\nabla\cdot\mathbf{u})\,\mathbf{I} + E + \Omega,$$

with $E = \frac{1}{2}\big(A + A^{\mathsf{T}}\big)$ and $\Omega = \frac{1}{2}\big(A - A^{\mathsf{T}}\big)$, where $A = \nabla\mathbf{u} - \frac{1}{3}(\nabla\cdot\mathbf{u})\,\mathbf{I}$.
Here $\nabla\cdot\mathbf{u} = \partial u/\partial x + \partial v/\partial y = a + (-a) = 0$, so $A = \nabla\mathbf{u}$ and

$$E = \frac{\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathsf{T}}}{2} = \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}, \qquad \Omega = \frac{\nabla\mathbf{u} - \nabla\mathbf{u}^{\mathsf{T}}}{2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

(Zero trace $\Rightarrow$ incompressible; E symmetric)

3) Vorticity.

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} = \big(0,\ 0,\ \partial v/\partial x - \partial u/\partial y\big) = (0,\ 0,\ 0).$$

4) Type of deformation. Since $\nabla\cdot\mathbf{u} = 0$ (no volumetric dilation), $\Omega = \mathbf{0}$ (no rigid-body rotation), and the only nonzero part is the symmetric $E = \mathrm{diag}(a, -a)$ with zero shear components, the motion is a *planar pure extension*: stretching along x at rate a and equal compression along y at rate a.

Question 7

Consider the (steady) 3D velocity field (units: x, y, z in m, t in s)

$$\mathbf{u}(x, y, z) = \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1.20\,x + 0.40\,y - 0.10\,z \\ -0.30\,x + 0.90\,y + 0.20\,z \\ 0.00\,x - 0.50\,y + 0.60\,z \end{pmatrix}.$$

1. Compute the velocity gradient tensor $\nabla\mathbf{u}$ and the divergence $\nabla\cdot\mathbf{u}$ (give numerical matrices/scalars).

2. Form the symmetric strain–rate tensor E and the antisymmetric rotation tensor Ω.

3. Compute the vorticity vector $\boldsymbol{\omega}$ (numerical values).

4. A small cubic fluid element of edge length $L_0 = 1\,\text{mm}$, centered at $(x_0, y_0, z_0) = (0, 0, 0)$, has initial edge vectors aligned with the axes:

$$\mathbf{e}_x = (L_0, 0, 0), \quad \mathbf{e}_y = (0, L_0, 0), \quad \mathbf{e}_z = (0, 0, L_0).$$

Using the linearized kinematics $\Delta\mathbf{e}_i \approx (\nabla\mathbf{u})\,\mathbf{e}_i\,\Delta t$ with $\Delta t = 0.02\,\text{s}$,

(a) compute the updated edge vectors $\mathbf{e}_i^{\text{new}} = \mathbf{e}_i + \Delta\mathbf{e}_i$,

(b) compute their new lengths,

(c) compute the angle (degrees) between $\mathbf{e}_x^{\text{new}}$ and $\mathbf{e}_y^{\text{new}}$.

5. Using $\dot{V}/V = \nabla\cdot\mathbf{u}$ for small times, estimate the element's volume after Δt and compare it to the volume obtained from the triple product $V^{\text{new}} \approx \mathbf{e}_x^{\text{new}} \cdot (\mathbf{e}_y^{\text{new}} \times \mathbf{e}_z^{\text{new}})$ (percent difference).

6. Classify the motion (dominantly extensional, shear, rotational?) and justify your answer using your E, Ω, and $\boldsymbol{\omega}$.

Solution

Given

$$\mathbf{u}(x, y, z) = \begin{pmatrix} 1.20\,x + 0.40\,y - 0.10\,z \\ -0.30\,x + 0.90\,y + 0.20\,z \\ -0.50\,y + 0.60\,z \end{pmatrix}, \qquad \nabla\mathbf{u} = \begin{bmatrix} 1.20 & 0.40 & -0.10 \\ -0.30 & 0.90 & 0.20 \\ 0.00 & -0.50 & 0.60 \end{bmatrix}.$$

1) Velocity gradient and divergence.

$$\nabla\cdot\mathbf{u} = \text{tr}(\nabla\mathbf{u}) = 1.20 + 0.90 + 0.60 = \boxed{2.70\ \text{s}^{-1}}.$$

2) Tensor decomposition: $\nabla\mathbf{u} = \frac{1}{3}(\nabla\cdot\mathbf{u})\mathbf{I} + E + \Omega$. First remove the isotropic (volumetric) part:

$$A \;=\; \nabla\mathbf{u} - \tfrac{1}{3}(\nabla\cdot\mathbf{u})\mathbf{I} \;=\; \begin{bmatrix} 0.30 & 0.40 & -0.10 \\ -0.30 & 0.00 & 0.20 \\ 0.00 & -0.50 & -0.30 \end{bmatrix}.$$

Then

$$E = \tfrac{1}{2}(A + A^{\mathsf{T}}) = \begin{bmatrix} 0.30 & 0.05 & -0.05 \\ 0.05 & 0.00 & -0.15 \\ -0.05 & -0.15 & -0.30 \end{bmatrix}, \qquad \Omega = \tfrac{1}{2}(A - A^{\mathsf{T}}) = \begin{bmatrix} 0 & 0.35 & -0.05 \\ -0.35 & 0 & 0.35 \\ 0.05 & -0.35 & 0 \end{bmatrix}.$$

3) Vorticity.

$$\boldsymbol{\omega} = \nabla\times\mathbf{u} = \begin{pmatrix} \partial w/\partial y - \partial v/\partial z \\ \partial u/\partial z - \partial w/\partial x \\ \partial v/\partial x - \partial u/\partial y \end{pmatrix} = \boxed{(-0.70,\ -0.10,\ -0.70)\ \text{s}^{-1}}.$$

4) Updated edge vectors, lengths, and angle (linearized over $\Delta t = 0.02\ s$). Let $L_0 = 1\,\text{mm} = 10^{-3}\,\text{m}$. With $\Delta\mathbf{e}_i \approx (\nabla\mathbf{u})\,\mathbf{e}_i\,\Delta t$:

$$\mathbf{e}_x^{\text{new}} = \boxed{(1.024\times10^{-3},\ -6.0\times10^{-6},\ 0)}\ \text{m},$$
$$\mathbf{e}_y^{\text{new}} = \boxed{(8.0\times10^{-6},\ 1.018\times10^{-3},\ -1.0\times10^{-5})}\ \text{m},$$
$$\mathbf{e}_z^{\text{new}} = \boxed{(-2.0\times10^{-6},\ 4.0\times10^{-6},\ 1.012\times10^{-3})}\ \text{m}.$$

New lengths (4 s.f.): $\|\mathbf{e}_x^{\text{new}}\| = 1.024\times10^{-3}$, $\|\mathbf{e}_y^{\text{new}}\| = 1.018\times10^{-3}$, $\|\mathbf{e}_z^{\text{new}}\| = 1.012\times10^{-3}$ m.

Angle between $\mathbf{e}_x^{\text{new}}$ and $\mathbf{e}_y^{\text{new}}$.

$$\cos\theta = \frac{\mathbf{e}_x^{\text{new}} \cdot \mathbf{e}_y^{\text{new}}}{\|\mathbf{e}_x^{\text{new}}\| \, \|\mathbf{e}_y^{\text{new}}\|} \qquad \text{so} \qquad \theta \approx \boxed{89.89^\circ}.$$

5) Volume change: divergence vs. triple product. Initial volume $V_0 = L_0^3 = 10^{-9}$ m^3.
We start from

$$\frac{\dot{V}}{V} = \nabla\cdot\mathbf{u},$$

that is,

$$\frac{dV}{dt} = (\nabla\cdot\mathbf{u})\, V.$$

For a small time step Δt, approximate the derivative using a forward finite difference:

$$\frac{dV}{dt} \;\approx\; \frac{V(t+\Delta t) - V(t)}{\Delta t}.$$

$$\frac{V(t+\Delta t) - V(t)}{\Delta t} \;\approx\; (\nabla\cdot\mathbf{u})\, V(t).$$

$$V(t+\Delta t) \;\approx\; V(t) + (\nabla\cdot\mathbf{u})\, V(t)\, \Delta t.$$

Express with initial volume $V_0 = V(0)$.

$$V_{\text{est}} \approx V_0\,(1 + (\nabla\cdot\mathbf{u})\,\Delta t) = 10^{-9}\,(1 + 2.70 \times 0.02) = \boxed{1.054 \times 10^{-9}\ \text{m}^3}.$$

Geometric check via triple product $V_{\text{tp}} \approx \mathbf{e}_x^{\text{new}} \cdot (\mathbf{e}_y^{\text{new}} \times \mathbf{e}_z^{\text{new}})$:

$$\boxed{V_{\text{tp}} \approx 1.05503 \times 10^{-9}\ \text{m}^3}, \qquad \text{percent difference} = \frac{V_{\text{tp}} - V_{\text{est}}}{V_{\text{est}}} \times 100\% \approx \boxed{0.10\%}.$$

6) Classification of the motion.

- $\nabla\cdot\mathbf{u} = 2.70 > 0$: net *dilation/expansion*.
- E has positive diagonals (1.20, 0.90, 0.60) and nonzero off-diagonals (± 0.05, -0.15): *extension* in all three axes plus *shear (angular) deformation*.
- $\Omega \neq 0$ and $\boldsymbol{\omega} = (-0.70, -0.10, -0.70)\ \text{s}^{-1}$ ($\|\boldsymbol{\omega}\| \approx 0.995\,\text{s}^{-1}$): finite *rigid-body rotation* component.

Overall, the motion is *dominantly extensional with positive volumetric dilation*, accompanied by *moderate shear* and *nonzero rotation*.

Exam-style questions

Question 8

Explain the difference between **vorticity** and **angular deformation (shear strain)** of a fluid element.
Can a fluid element:

1. have non-zero vorticity without undergoing angular deformation?
2. undergo angular deformation while having zero vorticity?

Briefly justify your answers.

Solution

Vorticity describes the local rigid-body rotation of a fluid element and is defined as

$$\boldsymbol{\omega} = \nabla \times \mathbf{u}.$$

Angular deformation describes the change in the angle between initially perpendicular material lines and is associated with the off-diagonal components of the strain-rate tensor,

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad i \neq j.$$

1. **Yes.** A rigid-body rotation has non-zero vorticity but no angular deformation.
2. **Yes.** A flow can have zero vorticity while still producing angular deformation. In this case, the rotational contributions cancel, while their symmetric contribution produces a change in the angle between material lines.

Question 9

Consider the two-dimensional velocity field

$$\mathbf{u}(x, y) = (ay, ax),$$

where a is a positive constant.

1. Calculate the divergence $\nabla \cdot \mathbf{u}$.
2. Calculate the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$.
3. Calculate the strain-rate tensor $\mathbf{S}$.
4. Describe how an initially square fluid element deforms.

Solution

The velocity components are

$$u = ay, \qquad v = ax.$$

The velocity gradient is

$$\nabla \mathbf{u} = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}.$$

The divergence is

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

The vorticity is

$$\boldsymbol{\omega} = \left(0, 0, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = (0, 0, a - a) = \mathbf{0}.$$

The strain-rate tensor is

$$\mathbf{S} = \frac{1}{2}\left[\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right] = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}.$$

Therefore, the flow has no volumetric deformation and no rigid-body rotation, but it has non-zero angular deformation. An initially square fluid element deforms into a rhombus, while its area remains unchanged to first order.