# Fluid Mechanics 4 (Mechanical)
# Tutorial week 1

## Question 1

Compute the derivative $dy/dx$ of the function $y = x \sin x$

### Solution

The product rule is used to solve $\frac{dy}{dx}$.
Product rule:

$$y = f(x) \cdot g(x)$$

$$\frac{dy}{dx} = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

where (') indicates the derivative with respect to $x$.
Let $f(x) = x$ and $g(x) = \sin(x)$, therefore:

$$f'(x) = 1 \quad \text{and} \quad g'(x) = \cos(x)$$

So,

$$\frac{dy}{dx} = 1 \cdot \sin(x) + x \cdot \cos(x) = \sin(x) + x \cdot \cos(x)$$

## Question 2

Consider the 2-dimensional velocity field that is described by the equation:

$$\boldsymbol{V} = 4xy\boldsymbol{e}_1 + 2(x^2 - y^2)\boldsymbol{e}_2$$

Where $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ represent the unit vectors in the $x$- and $y$-directions. Is this flow field irrotational (zero vorticity)?

### Solution

The flow is irrotational if $\nabla \times \boldsymbol{V} = 0$.

$$\nabla \times \boldsymbol{V} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\boldsymbol{e}_1 + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\boldsymbol{e}_2 + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\boldsymbol{e}_3$$

In a 2D flow field in the $x$-$y$ plane, the vorticity is in the $z$-direction ($\boldsymbol{e}_3$).
Thus,

$$\nabla \times \boldsymbol{V} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\boldsymbol{e}_3,$$

where $u = 4xy$ and $v = 2(x^2 - y^2)$.

$$\frac{\partial u}{\partial y} = 4x, \quad \frac{\partial v}{\partial x} = 4x;$$

$$\nabla \times \boldsymbol{V} = 4x - 4x = 0;$$

the flow is irrotational.

# Question 3

Demonstrate that the cross product is not commutative:

$$\boldsymbol{u} \times \boldsymbol{v} \neq \boldsymbol{v} \times \boldsymbol{u}$$

where $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\boldsymbol{v} = (v_1, v_2, v_3)$.

## Solution

$$\boldsymbol{u} \times \boldsymbol{v} = (u_2v_3 - v_2u_3)\boldsymbol{e}_1 + (u_3v_1 - v_3u_1)\boldsymbol{e}_2 + (u_1v_2 - v_1u_2)\boldsymbol{e}_3$$

$$\boldsymbol{v} \times \boldsymbol{u} = (v_2u_3 - u_2v_3)\boldsymbol{e}_1 + (v_3u_1 - u_3v_1)\boldsymbol{e}_2 + (v_1u_2 - u_1v_2)\boldsymbol{e}_3$$

Hence:

$$\boldsymbol{u} \times \boldsymbol{v} \neq \boldsymbol{v} \times \boldsymbol{u},$$

and is therefore not commutative.

# Question 4

Consider a Cartesian coordinate system with velocity components:

$$u = x^2 + y^2 + z^2$$

$$v = xy + yz + z$$

$$w = ?$$

Determine the form of the $w$ velocity component if the flow has zero divergence.

## Solution

If the flow has zero divergence:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial v}{\partial y} = x + z$$

Thus:

$$2x + x + z + \frac{\partial w}{\partial z} = 0$$

Simplifying the equation:

$$3x + z + \frac{\partial w}{\partial z} = 0$$

Isolating $\frac{\partial w}{\partial z}$:

$$\frac{\partial w}{\partial z} = -(3x + z)$$

Integrating both sides with respect to $z$:

$$w(z) = \int -(3x + z)\, dz$$

This results in:

$$w(z) = -3xz - \frac{z^2}{2} + C(x, y)$$

Where $C(x, y)$ is an arbitrary function of $x$ and $y$, as the constant of integration can depend on $x$ and $y$.

# Question 5

Consider the scalar field

$$\phi(x, y, z) = x^2 y + yz,$$

and the vector field

$$\mathbf{F}(x, y, z) = (xy,\ y^2 z,\ xz).$$

1. Compute the **gradient** of $\phi(x, y, z)$.
2. Compute the **divergence** of $\mathbf{F}(x, y, z)$.
3. Compute the **curl** of $\mathbf{F}(x, y, z)$.

## Solution

**1) Gradient of $\phi$.**

$$\nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) = \big(2xy,\ x^2 + z,\ y\big).$$

**2) Divergence of F.**

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(y^2 z) + \frac{\partial}{\partial z}(xz) = y + 2yz + x = x + y + 2yz.$$

**3) Curl of F.**

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & y^2 z & xz \end{vmatrix} = \left(\frac{\partial(xz)}{\partial y} - \frac{\partial(y^2 z)}{\partial z},\ \frac{\partial(xy)}{\partial z} - \frac{\partial(xz)}{\partial x},\ \frac{\partial(y^2 z)}{\partial x} - \frac{\partial(xy)}{\partial y}\right).$$

Thus

$$\nabla \times \mathbf{F} = \big(0 - y^2,\ 0 - z,\ 0 - x\big) = \boxed{(-y^2,\ -z,\ -x)}.$$

# Question 6

Let the velocity field be

$$\mathbf{u} = (u, v, w).$$

1. Write the divergence $\nabla \cdot \mathbf{u}$ in index notation.
2. Write the convective term $(\mathbf{u} \cdot \nabla)\mathbf{u}$ in index notation.

## Solution

Assume Cartesian coordinates with $x_1 = x$, $x_2 = y$, $x_3 = z$ and velocity components $u_1 = u$, $u_2 = v$, $u_3 = w$. We use Einstein summation over repeated indices.

**1) Divergence.**

$$\frac{\partial u_i}{\partial x_i} \;=\; \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}.$$

**2) Convective term.** The term $(\mathbf{u} \cdot \nabla)\mathbf{u}$ is

$$u_j \frac{\partial u_i}{\partial x_j} = \left( u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z},\; u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z},\; u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \right).$$

# Question 7

The transport of a passive scalar, often denoted by $\phi$, can be described by the advection-diffusion equation. The general form of this equation is given by:

$$\frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \nabla \phi = \nabla \cdot (D \nabla \phi) + S$$

Where:

- $\phi$ is the passive scalar (e.g., concentration of a pollutant).
- $t$ is time.
- $\mathbf{u}$ is the velocity field (advection).
- $D$ is the diffusion coefficient (representing molecular diffusion).
- $S$ is a source term (representing any additional sources or sinks of the scalar).

Write this equation in index form.

## Solution

In index notation using Einstein summation convention, the advection-diffusion equation for a passive scalar can be expressed as follows:

$$\frac{\partial \phi}{\partial t} + u_i \frac{\partial \phi}{\partial x_i} = \frac{\partial}{\partial x_j} \left( D \frac{\partial \phi}{\partial x_j} \right) + S$$

# Question 8

Use the **divergence theorem** to compute the outward flux of the vector field

$$\mathbf{F}(x, y, z) = (x^2,\, y^2,\, z^2)$$

through the surface of the cube bounded by

$$x = 0,\ x = 1, \qquad y = 0,\ y = 1, \qquad z = 0,\ z = 1.$$

## Solution

By the divergence theorem,

$$\iint_{\partial V} \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_V \nabla \cdot \mathbf{F}\, dV,$$

where $V = [0, 1]^3$ and $\mathbf{F}(x, y, z) = (x^2, y^2, z^2)$.
Compute the divergence:

$$\nabla \cdot \mathbf{F} = 2x + 2y + 2z.$$

Therefore,

$$\iint_{\partial V} \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_{[0,1]^3} (2x + 2y + 2z)\, dV = 2\left( \iiint x\, dV + \iiint y\, dV + \iiint z\, dV \right).$$

Over the unit cube,

$$\iiint_{[0,1]^3} x\, dV = \int_0^1 \int_0^1 \int_0^1 x\, dx\, dy\, dz = \frac{1}{2},$$

and similarly $\iiint y\, dV = \frac{1}{2}$ and $\iiint z\, dV = \frac{1}{2}$.
Hence,

$$\iint_{\partial V} \mathbf{F} \cdot \mathbf{n}\, dS = 2\left( \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \right) = 2 \cdot \frac{3}{2} = \boxed{3}.$$

**Direct check via face integrals.** Compute $\iint_{\partial V} \mathbf{F} \cdot \mathbf{n}\, dS$ by summing the six faces.
*Faces $x = const.$* For $x = 1$, $\mathbf{n} = (1, 0, 0)$ and $\mathbf{F} \cdot \mathbf{n} = x^2 = 1$.

$$\iint_{x=1} \mathbf{F} \cdot \mathbf{n}\, dS = \int_0^1 \int_0^1 1\, dy\, dz = 1.$$

For $x = 0$, $\mathbf{n} = (-1, 0, 0)$ and $\mathbf{F} \cdot \mathbf{n} = -x^2 = 0$, so the flux is 0.
*Faces $y = const.$* For $y = 1$, $\mathbf{n} = (0, 1, 0)$ and $\mathbf{F} \cdot \mathbf{n} = y^2 = 1$, hence flux $= 1$. For $y = 0$, $\mathbf{n} = (0, -1, 0)$, $\mathbf{F} \cdot \mathbf{n} = -y^2 = 0$, flux $= 0$.
*Faces $z = const.$* For $z = 1$, $\mathbf{n} = (0, 0, 1)$ and $\mathbf{F} \cdot \mathbf{n} = z^2 = 1$, hence flux $= 1$. For $z = 0$, $\mathbf{n} = (0, 0, -1)$, $\mathbf{F} \cdot \mathbf{n} = -z^2 = 0$, flux $= 0$.
Summing all faces:

$$1 + 0 + 1 + 0 + 1 + 0 = \boxed{3},$$

which agrees with the divergence theorem result.

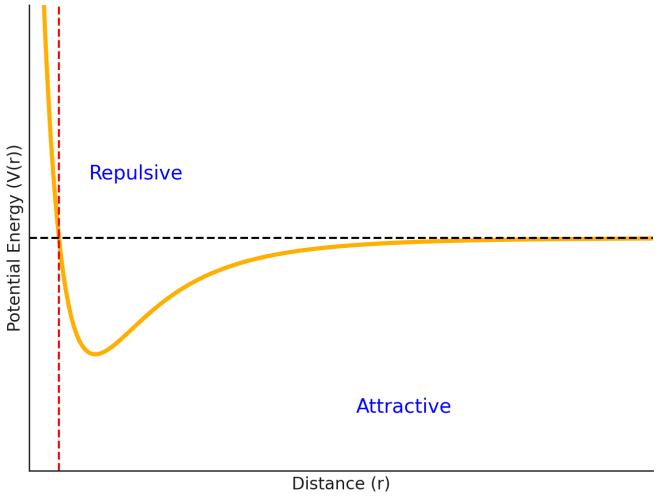
# Exam-style questions

# Question 9

Sketch the Lennard–Jones potential as a function of intermolecular distance, and indicate the region corresponding to gas-like behaviour.

### Solution

The *gas-like region* corresponds to large intermolecular distances $r$, where the attractive part of the Lennard–Jones potential is too weak to overcome the thermal oscillations of the molecules/atoms.




## Question 10

What is the fundamental difference between a solid and a fluid? Consider how each material responds when subjected to a constant shear stress.

### Solution

A **solid** can sustain a shear stress while remaining at rest. When a constant shear stress is applied, the solid deforms until it reaches a new equilibrium configuration. For an elastic solid, the shear stress is related to the deformation, for example,

$$\tau = G\gamma,$$

where $G$ is the shear modulus and $\gamma$ is the shear strain.
A **fluid**, in contrast, cannot sustain a shear stress while at rest. Any non-zero shear stress causes the fluid to deform continuously, i.e. to flow. For a Newtonian fluid, the shear stress is proportional to the *rate* of deformation,

$$\tau = \mu \frac{du}{dy},$$

where $\mu$ is the dynamic viscosity.
Therefore, the fundamental distinction is that, under a constant shear stress, a solid reaches a finite deformation, whereas a fluid continues to deform for as long as the shear stress is applied.