Fluid Mechanics 4 (Mechanical)
Tutorial week 4

Question 1

Consider a fluid flowing through a cylindrical pipe of varying diameter. The velocity field of the fluid is given by:

$$\mathbf{u}(x, r) = (v(x, r), 0, 0)$$

where $v(x, r)$ is the axial velocity, which varies with the axial x and radial r positions. The mass flow rate $\dot{m}$ at any cross-section of the pipe can be expressed as:

$$\dot{m} = \int_A \rho \mathbf{u} \cdot \mathbf{n} \, dA$$

where ρ is the fluid density and A is the cross-sectional area.

- Derive the expression for mass conservation for an incompressible fluid with constant density in a control volume that includes a segment of the pipe with varying diameter.
- If the diameter of the pipe at section 1 is D_1 and at section 2 is D_2, and the fluid is incompressible with constant density ρ, express the relationship between the velocities v_1 and v_2 at these two sections, assuming that the velocities v_1 and v_2 are spatially homogeneous over the sections (independent of r).

Question 2

A rigid, well-mixed tank of volume $V_{\text{tank}} = 2$ m^3 contains a gas initially at uniform density $\rho_0 = 1$ kg/m^3. A feed line delivers gas at a constant *mass* flow rate $\dot{m}_{\text{in}} = 0.2$ kg/s. Gas exits through a valve whose *volumetric* outflow rate is constant, $\dot{V}_{\text{out}} = 0.1$ m^3/s. Because the tank is perfectly mixed, the density of the outflow equals the instantaneous tank density, $\rho(t)$. Assume single phase, no leaks, and spatially uniform properties in the tank.

1. Write the **Reynolds Transport Theorem (RTT)** for mass and derive the ODE (Ordinary Differential Equation) for the tank mass $m(t)$.
2. Solve the equation derived in the previous point to obtain $m(t)$ and $\rho(t)$ given $m(0) = \rho_0 V_{\text{tank}}$. The equation derived in the previous point is not easy to solve. If you do not remember how to solve it, the solution is:

   $$m(t) = m_0 \, e^{-kt} + \frac{\dot{m}_{\text{in}}}{k}\Big(1 - e^{-kt}\Big).$$

   with $k \equiv \dot{V}_{\text{out}}/V_{\text{tank}}$.

   While the math here is more complex than typical exam questions, this exercise is useful to understand the link between the conservation equation and a real problem.
3. If the gas is ideal at constant $T = 300$ K with $R = 287$ J/(kg K), give $p(t)$.

Question 3

Consider a fluid with density $\rho(\mathbf{x}, t)$ and velocity field $\mathbf{u}(\mathbf{x}, t)$.

(a) Starting from the **integral mass balance** over a fixed control volume V_0 with control surface S_0,

$$\int_{V_0} \frac{\partial \rho}{\partial t}\, dV + \int_{S_0} \rho\, \mathbf{u} \cdot \mathbf{n}\, dS = 0,$$

use the divergence theorem to derive the **differential form** of mass conservation. State the mathematical conditions under which your steps are valid.

(b) Expand your result to obtain $\dfrac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{u} = 0$, and rewrite it using the material derivative D/Dt. Briefly interpret each term.

(c) If $\rho = \rho_0 = \text{const}$ in space and time, show that $\nabla \cdot \mathbf{u} = 0$.

(d) Check incompressibility for the follwoing flows.

(i) $\mathbf{u}(x, y, z) = (ax,\, by,\, cz)$. Find the condition on a, b, c for incompressibility.

(ii) $\mathbf{u}(x, y) = (\alpha x,\, -\alpha y)$ (planar stagnation flow). Is it incompressible?

(iii) $\mathbf{u}(x, y) = \left(-\dfrac{\Gamma}{2\pi} \dfrac{y}{r^2},\, \dfrac{\Gamma}{2\pi} \dfrac{x}{r^2} \right)$, where $r^2 = x^2 + y^2$ (potential vortex). Evaluate $\nabla \cdot \mathbf{u}$ for $r \neq 0$.

Exam-style questions

Question 4

The density in a horizontal flow $u = U(y, z)\, \boldsymbol{e}_x$ is given by $\rho(x, y, z) = f(x - Ut, y, z)$, where $f(x, y, z)$ is the density distribution at $t = 0$. Is this flow incompressible?