Fluid Mechanics 4

Antonio Attili

School of Engineering, University of Edinburgh

antonio.attili@ed.ac.uk

Lecture 9

Applications of Navier-Stokes Equations:
Analytical solutions for laminar steady flows in simple geometries.

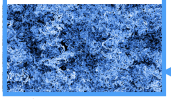


Roadmap

First principles:
$F = ma$
Mass conservation
Constitutive relations

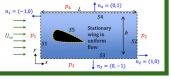
NAVIER STOKES EQUATIONS (+CONTINUITY)
⇒impossible to solve analytically

TURBULENCE
Turbulence theory
Law of the Wall



If $Re \uparrow$ the solution becomes unstable (turbulent)

Global quantities (lift, drag)



Integral equations

Hypotheses and observation to simplify the equations

Bernoulli relation between two points in the field

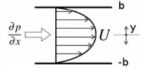
$$p_1 + \frac{1}{2}\rho V_1^2 + \rho g h_1 = p_2 + \frac{1}{2}\rho V_2^2 + \rho g h_2$$

Steady, inviscid, incompressible…

Simple geometries

$\mu = 0$
…

Analytical solutions of NS ⇒ solution of the velocity field

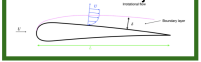


Ideal flows
Potential flows
⇒ solution of the velocity field



Close to a wall or object $\mu \neq 0$

Boundary layer theory close to the object + Potential flow far from the object



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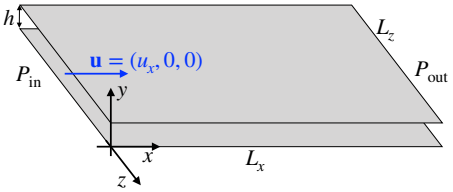
Analytical Solutions of Navier–Stokes Equations

- In some simple flow configurations, the **Navier–Stokes equations** can be solved exactly.
- These solutions are invaluable to:
  - build physical intuition,
  - validate numerical methods,
  - and provide benchmarks for experiments.
- Examples include:
  - **Poiseuille flow** – pressure-driven flow between parallel plates,
  - **Couette flow** – shear-driven flow between moving plates,
  - **Couette–Poiseuille flow** – combined pressure and shear driven.
- Despite their simplicity, these flows capture the essential balance between **pressure forces**, **viscous stresses**, and **boundary conditions**.

The Poisseuille Flow

Configuration

- Consider steady, incompressible, fully-developed laminar flow between two parallel, infinite plane walls.
- The flow is driven by a constant pressure gradient in the x-direction.
- The goal is to derive the velocity profile $u(y)$ from the incompressible Navier-Stokes equations.



Assumptions

- Steady flow: $\frac{\partial}{\partial t} = 0$.
- Incompressible: $\nabla \cdot \mathbf{u} = 0$.
- Fully developed: $u = u(y)$, i.e., velocity depends only on y.
- Flow driven by pressure gradient $\frac{\partial p}{\partial x} = -G$, where G is constant.
- No-slip boundary condition: velocity is zero at both walls.
- The bottom wall is at $y = 0$ and the top wall is at $y = h$.
- Gravity not considered. With gravity, the solution would be the same, but the pressure would depend on y.

The Navier-Stokes equation for steady, incompressible flow, with no body forces (e.g., gravity), are given by:

$$\rho\left(\mathbf{u}\cdot\nabla\mathbf{u}\right)=-\nabla p+\mu\nabla^2\mathbf{u}$$

In two-dimensions, in component form, these are:

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

Since the flow is fully developed and varies only in the y-direction:

- $\frac{\partial u}{\partial x} = 0$ (velocity profile does not change along x).
- $v = 0$ (no velocity in the y-direction).
- $\frac{\partial p}{\partial y} = 0$ (pressure varies only in x).

The x-momentum equation simplifies to:

$$0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2}$$

With $\frac{\partial p}{\partial x} = -G$, the equation becomes:

$$\mu \frac{d^2 u}{dy^2} = -G$$

Solving the Differential Equation

The equation to solve is:

$$\frac{d^2u}{dy^2} = -\frac{G}{\mu}$$

Integrating once with respect to y:

$$\frac{du}{dy} = -\frac{G}{\mu}y + C_1$$

Integrating again with respect to y:

$$u(y) = -\frac{G}{2\mu}y^2 + C_1y + C_2$$

Boundary Conditions

Apply the no-slip boundary conditions:

- At $y = 0$, $u(0) = 0$ (no slip at the bottom wall).
- At $y = h$, $u(h) = 0$ (no slip at the top wall).

Substituting $y = 0$ into the velocity equation:

$$u(0) = 0 = -\frac{G}{2\mu}(0)^2 + C_1(0) + C_2 \Rightarrow C_2 = 0$$

Substituting $y = h$ into the velocity equation:

$$u(h) = 0 = -\frac{G}{2\mu}h^2 + C_1 h \Rightarrow C_1 = \frac{Gh}{2\mu}$$

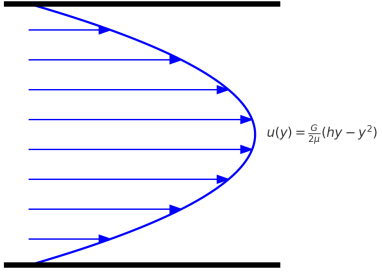
Velocity Profile

Substituting the values of C_1 and C_2 into the equation for $u(y)$:

$$u(y) = -\frac{G}{2\mu}y^2 + \frac{Gh}{2\mu}y$$

Factor out $\frac{G}{2\mu}$:

$$u(y) = \frac{G}{2\mu}\left(hy - y^2\right)$$



Generalisation: Couette-Poiseuille flow

Configuration

- Consider steady, incompressible, fully-developed flow between two parallel, infinite plane walls.
- The flow is driven by a constant pressure gradient in the x-direction.
- The top plate moves with a constant velocity U in the x-direction.
- The goal is to derive the velocity profile $u(y)$ from the incompressible Navier-Stokes equations.

Assumptions

- Steady flow: $\frac{\partial}{\partial t} = 0$.
- Incompressible: $\nabla \cdot \mathbf{u} = 0$.
- Fully developed: $u = u(y)$, i.e., velocity depends only on y.
- Flow driven by pressure gradient $\frac{\partial p}{\partial x} = -G$, where G is constant.
- The bottom wall (at $y = 0$) is stationary, and the top wall (at $y = h$) moves with velocity U.

Boundary Conditions and Final Velocity Profile

The boundary condition on the top wall is different this time:

- At $y = 0$, the bottom wall is stationary: $u(0) = 0$.
- At $y = h$, the top wall moves with velocity U: $u(h) = U$.

Substituting $y = 0$ into the velocity equation:

$$u(0) = 0 \Rightarrow C_2 = 0$$

Substituting $y = h$ into the velocity equation:

$$u(h) = U \Rightarrow C_1 = \frac{U + \frac{Gh^2}{2\mu}}{h}$$

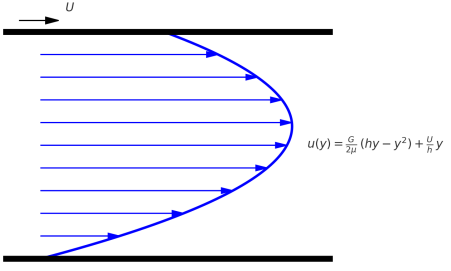
Velocity Profile

Substituting the values of C_1 and $C_2 = 0$ into the general solution for $u(y)$:

$$u(y) = -\frac{G}{2\mu}y^2 + \left(\frac{U + \frac{Gh^2}{2\mu}}{h}\right)y$$

Simplifying:

$$u(y) = \frac{G}{2\mu}\left(hy - y^2\right) + \frac{U}{h}y$$



Summary

Poisseuille flow

$$u(y) = \frac{G}{2\mu}\left(hy - y^2\right)$$

Couette flow

$$u(y) = \frac{U}{h}y$$

Couette-Poiseuille flow

$$u(y) = \frac{G}{2\mu}\left(hy - y^2\right) + \frac{U}{h}y$$