Fluid Mechanics 4

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Lecture 8

Constitutive Laws and the Navier-Stokes Equation



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Conservation of Momentum

- Integral form of the conservation of momentum for a control volume V_0 with surface S_0

$$\int_{V_0} \frac{\partial \rho \boldsymbol{u}}{\partial t}\, dV + \int_{S_0} \rho \boldsymbol{u}\boldsymbol{u} \cdot \boldsymbol{n}\, dS = -\int_{S_0} p\boldsymbol{I} \cdot \boldsymbol{n}\, dS + \int_{S_0} \boldsymbol{\tau} \cdot \boldsymbol{n}\, dS + \int_{V_0} \rho \boldsymbol{f}\, dV,$$

where we defined:

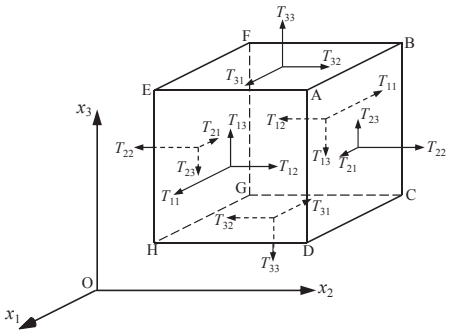
$$\boldsymbol{T} = -p\boldsymbol{I} + \boldsymbol{\tau}$$

- In differential forms:

$$\frac{\partial(\rho \boldsymbol{u})}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}\boldsymbol{u}) = -\nabla p + \nabla \cdot \boldsymbol{\tau} + \rho \boldsymbol{f}$$

$$\rho \frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + \rho f_i$$

Constitutive equation for a Newtonian Fluid: The Stress Tensor



- $\boldsymbol{T}$ is stress tensor, which can be written as T_{ij} in index form.
- First index of T_{ij} is the direction of the normal to the surface on which the stress is considered
- Second index of T_{ij} indicates the direction in which the stress acts
- The diagonal elements T_{11}, T_{22}, and T_{33} of the stress matrix are the normal stresses, and the off-diagonal elements are the tangential or shear stresses
- For a volume that shrink to a point, using consideration about the torque that the stress should impose, it is possible to show that the **stress tensor is symmetric**:

$$T_{ij} = T_{ji}$$

therefore the stress tensor has only six independent components (note: the symmetry is sometimes violated in specific circumstances, for example for polarised molecules)

Constitutive equation for a Newtonian Fluid: Isotropic Stress

- The relationship between the stress and deformation in a continuum is called a constitutive equation
  - **Linear constitutive equation** between stress T_{ij} and $\partial u_i/\partial x_j$ is examined here
  - A fluid that follows the simplest possible linear constitutive equation is known as a **Newtonian fluid**

- Isotropic stress
  - In a fluid at rest, there are only normal components of stress on a surface, and the stress does not depend on the orientation of the surface; the stress is **isotropic**.
  - The only second-order isotropic tensor is the Kronecker delta, δ_{ij}. Therefore, the stress in a static fluid must be of the form:

$$T_{ij} = -p\delta_{ij}$$

where p is the **thermodynamic pressure** related to ρ and $\mathcal{T}$ (temperature) by an equation of state such as that for a perfect gas $p = \rho R\mathcal{T}$
(note: The negative sign occurs because the normal components of T are regarded as positive if they indicate tension rather than compression)

Constitutive equation for a Newtonian Fluid: Deviatoric Stress

- A moving fluid develops additional stress components, τ_{ij}, because of viscosity
- These stress components appear as **both diagonal and off-diagonal** components

$$T_{ij} = -p\delta_{ij} + \tau_{ij}$$

The *fluid-dynamic contribution* τ_{ij} to the stress tensor is called the **deviatoric stress tensor**

- Since it must be invariant under Galilean transformations, τ cannot depend on the absolute fluid velocity so it must depend on the velocity gradient tensor $\partial u_i/\partial x_j$
- Stress can develop only for fluid elements that change shape, so we are interested only in the symmetric part of the velocity gradient, the strain rate tensor $S_{ij} = 1/2(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$
- Most general form:

$$\tau_{ij} = \mathcal{K}_{ijmn} S_{mn}$$

where $\mathcal{K}_{ijmn}$ is a fourth-order tensor having 81 components

- However, this level of generality is unnecessary when the stress tensor is symmetric, and the fluid is isotropic, so let's simplify.

Constitutive equation for a Newtonian Fluid: Deviatoric Stress

- In an isotropic fluid, $\mathcal{K}_{ijmn}$ must be isotropic, so it can be written as:

$$K_{ijmn} = \lambda\delta_{ij}\delta_{mn} + \mu\delta_{im}\delta_{jn} + \gamma\delta_{in}\delta_{jm}$$

  where where λ, μ, and γ are scalars that in general depend on the local thermodynamic state.
- If we also require that τ_{ij} is symmetric, then it must be

$$\gamma = \mu$$

- So we can write

$$\tau_{ij} = 2\mu S_{ij} + \lambda S_{mm}\delta_{ij}$$

  where $S_{mm} = \nabla \cdot \boldsymbol{u}$ is the volumetric strain rate.
- The complete stress tensor is then:

$$T_{ij} = -p\delta_{ij} + 2\mu S_{ij} + \lambda S_{mm}\delta_{ij}$$

  - μ is called the first coefficient of viscosity or shear viscosity (usually just called "viscosity")
  - λ is the second coefficient of viscosity

Constitutive equation for a Newtonian Fluid: Deviatoric Stress

- The two scalars μ and λ can also be related.
- Setting $i = j$ and summing (note $\delta_{ii} = 3$), we have:

$$T_{ii} = -3p + (2\mu + 3\lambda) S_{mm}$$

- So we can express the pressure as:

$$p = -\frac{1}{3} T_{ii} + \left(\frac{2}{3}\mu + \lambda \right) \nabla \cdot \mathbf{u}$$

- We can define a *mean pressure* (as opposed to thermodynamic pressure p):

$$\bar{p} \equiv -\frac{1}{3} T_{ii}$$

and write:

$$p - \bar{p} = \left(\frac{2}{3}\mu + \lambda \right) \nabla \cdot \mathbf{u}$$

- For a completely incompressible fluid we can only define a **mechanical or mean pressure**, because there is no equation of state to determine a thermodynamic pressure.
- The **absolute pressure in an incompressible fluid is indeterminate**, and **only its gradients can be determined** from the equations of motion.

Constitutive equation for a Newtonian Fluid: Incompressible vs Compressible

- When $S_{ii} = \nabla \cdot \boldsymbol{u} = 0$:

$$p = \bar{p}$$

and

$$T_{ij} = -p\delta_{ij} + 2\mu S_{ij}$$

where p can only be interpreted as the mean (mechanical) pressure experienced by the fluid.

- More in general, for compressible flows, introducing the so-called **coefficient of bulk viscosity** $\mu_v = \lambda + 2\mu/3$:

$$T_{ij} = -p\delta_{ij} + \tau_{ij} = -p\delta_{ij} + 2\mu\left(S_{ij} - \frac{1}{3}S_{mm}\delta_{ij}\right) + \mu_v S_{mm}\delta_{ij}.$$

- However, also for compressible flows, we often use the **Stokes assumption**:

$$\mu_v = \lambda + 2\mu/3 = 0$$

Stress Tensor

$$\mathbf{T} = \begin{pmatrix} -p + 2\mu\frac{\partial u}{\partial x} + \left(\mu_v - \frac{2}{3}\mu\right)\nabla\cdot\mathbf{u} & \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \\ \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & -p + 2\mu\frac{\partial v}{\partial y} + \left(\mu_v - \frac{2}{3}\mu\right)\nabla\cdot\mathbf{u} & \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) & -p + 2\mu\frac{\partial w}{\partial z} + \left(\mu_v - \frac{2}{3}\mu\right)\nabla\cdot\mathbf{u} \end{pmatrix}$$

- Substituting the stress tensor definition in the momentum equation:

$$\rho\left(\frac{\partial u_j}{\partial t}+u_i\frac{\partial u_j}{\partial x_i}\right)=-\frac{\partial p}{\partial x_j}+\rho f_j+\frac{\partial}{\partial x_i}\left[\mu\left(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j}\right)+\left(\mu_v-\frac{2}{3}\mu\right)\frac{\partial u_m}{\partial x_m}\delta_{ij}\right]$$

  This is the **Navier-Stokes momentum equation**

- If we use the Stokes assumption $\mu_v=\lambda+2\mu/3=0$, then:

$$\rho\left(\frac{\partial u_j}{\partial t}+u_i\frac{\partial u_j}{\partial x_i}\right)=-\frac{\partial p}{\partial x_j}+\rho f_j+\frac{\partial}{\partial x_i}\left[\mu\left(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j}\right)-\frac{2}{3}\mu\frac{\partial u_m}{\partial x_m}\delta_{ij}\right]$$

Governing Equations of Fluids

- With the following hypotheses:
  - Continuum hypothesis
  - Fields are regular enough (continuous, differentiable, ...)
  - Newtonian fluid
  - Stokes assumption

- Conservation of mass (continuity):

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0$$

- Conservation of Momentum (Navier-Stokes Equation):

$$\rho \left(\frac{\partial u_j}{\partial t} + u_i \frac{\partial u_j}{\partial x_i} \right) = -\frac{\partial p}{\partial x_j} + \rho f_j + \frac{\partial}{\partial x_i} \left[\mu \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) - \frac{2}{3} \mu \frac{\partial u_m}{\partial x_m} \delta_{ij} \right]$$

- These are 4 equations with 5 unknowns (3 component of velocity, pressure, density). They must be completed with additional equations, which can be different for different cases.
  - Equation of state linking density and pressure, for example $p = \rho R \mathcal{T}$ (where R is the gas constant and $\mathcal{T}$ the temperature).
  - However, we introduced temperature now, so we need another equation for it, for example the equation of energy conservation.

Special Cases of Navier-Stokes: Incompressible Flows

- For an incompressible flow with constant density, we have
- Mass equation

$$\nabla \cdot \boldsymbol{u} = \frac{\partial u_m}{\partial x_m} = 0$$

- Navier Stokes equation

$$\rho\left(\frac{\partial u_j}{\partial t} + u_i \frac{\partial u_j}{\partial x_i}\right) = -\frac{\partial p}{\partial x_j} + \rho f_j + \frac{\partial}{\partial x_i}\left[\mu\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)\right]$$

- if the viscosity is assumed to be constant (no significant variation of temperature and composition):

$$\boxed{\rho\left(\frac{\partial u_j}{\partial t} + u_i \frac{\partial u_j}{\partial x_i}\right) = -\frac{\partial p}{\partial x_j} + \rho f_j + \mu \frac{\partial^2 u_j}{\partial x_i \partial x_i}}$$

- 4 equations with 4 unknowns. They can be solved!
- Incompressible flows:
  - Most liquid flows
  - Gas flows with small Mach number $Ma = U/c < 0.3$
  - Speed of sound $c = 330\, m/s$ for air at ambient conditions
- **This form of the NS equations will be used widely in the rest of the course.**

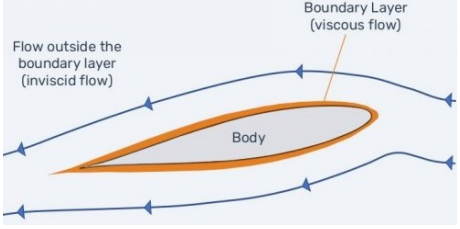
Special Cases of Navier-Stokes: Euler Equation

- Inviscid flows: viscous forces are neglected.
- Conservation of mass (continuity):

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0$$

- Conservation of Momentum (Euler Equation):

$$\rho \left(\frac{\partial u_j}{\partial t} + u_i \frac{\partial u_j}{\partial x_i} \right) = -\frac{\partial p}{\partial x_j} + \rho f_j$$



- These are still 4 equations with 5 unknowns.
- Equation of state and energy equation usually included to close the system.

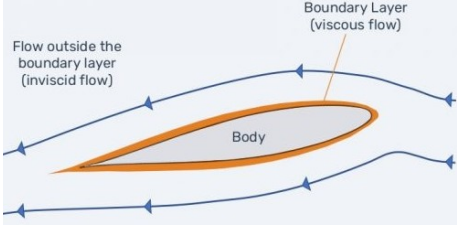
Special Cases of Navier-Stokes: Inviscid and Incompressible

- Inviscid flows: viscous forces are neglected.
- Incompressible: constant density and zero divergence
- Conservation of mass (continuity):

$$\frac{\partial u_m}{\partial x_m} = 0$$

- Conservation of Momentum:

$$\rho\left(\frac{\partial u_j}{\partial t} + u_i\frac{\partial u_j}{\partial x_i}\right) = -\frac{\partial p}{\partial x_j} + \rho f_j$$



- 4 equations with 4 unknowns. They can be solved!

Special Cases of Navier-Stokes: Incompressible and Irrotational

- Incompressible: constant density and zero divergence
- Irrotational: no vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{u} = 0$
- Conservation of mass (continuity):

$$\frac{\partial u_m}{\partial x_m} = 0$$

- Conservation of Momentum:

$$\rho \left(\frac{\partial u_j}{\partial t} + u_i \frac{\partial u_j}{\partial x_i} \right) = -\frac{\partial p}{\partial x_j} + \rho f_j$$

- 4 equations with 4 unknowns.
- Why are viscous terms removed?
  - If fluid is moving without rotating, it will not rotate due to body or pressure forces. Development of rotation requires a shear stress (i.e., angular deformation).
  - Without viscous forces, fluid will not start rotating
- **This will be the starting point to study potential flows.**