Fluid Mechanics 4

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Lecture 7

Conservation Laws



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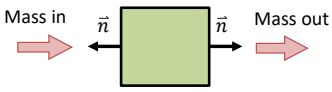
Conservation of Mass - Integral form

- For a material system contained in the material volume $V(t)$, neglecting nuclear reactions and relativistic effects, the mass M does not change in time

$$\frac{dM}{dt} = 0 \Rightarrow \frac{d}{dt}\int_{V(t)} \rho \, dV = 0$$

- We can use the Reynolds Transport Theorem, with $B = M$ and therefore $b = 1$, written for the control volume V_0

$$\int_{V_0} \frac{\partial \rho}{\partial t} \, dV + \int_{S_0} \rho \boldsymbol{u} \cdot \boldsymbol{n} \, dS = 0.$$



- If V_0 is fixed and the conditions for the divergence theorem are satisfied (continuity of the functions, etc):

$$\int_{V_0} \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) \right) dV = 0$$

- The last equation in the previous slide:

$$\int_{V_0} \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) \right) dV = 0$$

must be satisfied for any choice of the volume V_0, so the only possibility is the following property for the integrand function:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0$$

- This can be expanded as

$$\frac{\partial \rho}{\partial t} + \boldsymbol{u} \cdot \nabla \rho + \rho \nabla \cdot \boldsymbol{u} = 0$$

- If $\rho(\boldsymbol{x}, t) = const$ (for example for an incompressible flow) then:

$$\nabla \cdot \boldsymbol{u} = 0,$$

Conservation of Momentum - Integral form

- Newton's second law can be written as:

$$\boldsymbol{F} = m\boldsymbol{a}$$

  or using the momentum $\boldsymbol{Q}$, as:

$$\boldsymbol{F} = \frac{d\boldsymbol{Q}}{dt}$$

- For a fluid, we can define:

$$\boldsymbol{Q} = \int_{V(t)} \rho \boldsymbol{u} \, dV$$

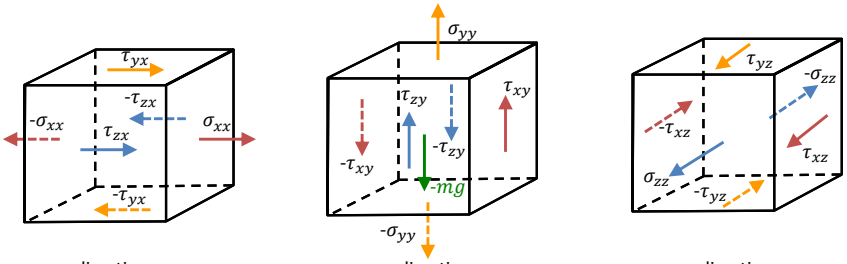
- With $\boldsymbol{F}$, we indicate all the forces that act on the volume:
  - $\boldsymbol{F}_S$: contact forces, acting on the surface of the volume, such as pressure and viscous forces.
  - $\boldsymbol{F}_V$: volume forces, such as weight or Coriolis force.
- Among the contact forces, we can separate pressure from other forces (such as friction):

$$\boldsymbol{F}_S = -\int_{S_0} p \, \boldsymbol{n} \, dS + \boldsymbol{F}'_S$$

- Using the Reynolds transport theorem and the above definition of $\boldsymbol{Q}$, we obtain:

$$\int_{V_0} \frac{\partial \rho \boldsymbol{u}}{\partial t} \, dV + \int_{S_0} \rho \boldsymbol{u}\boldsymbol{u} \cdot \boldsymbol{n} \, dS = -\int_{S_0} p \, \boldsymbol{n} \, dS + \boldsymbol{F}'_S + \boldsymbol{F}_V$$

Forces acting on a fluid volume



- Let's define:

$$\boldsymbol{F}_S = \int_{S_0} \boldsymbol{T} \cdot \boldsymbol{n}\, dS \quad \text{and} \quad \boldsymbol{F}_V = \int_{V_0} \rho \boldsymbol{f}\, dV$$

where $\boldsymbol{f}$ is the force for unit mass (it would be the gravity acceleration $\boldsymbol{g}$ for the weight force)
$\boldsymbol{T}$ is the surface stress tensor.

- Here, we also assume the following result that will be discussed later:

$$\boldsymbol{T} = -p\boldsymbol{I} + \boldsymbol{\tau}$$

where p is the pressure
$\boldsymbol{I}$ is the identity tensor
$\boldsymbol{\tau}$ is therefore the deviatoric part of the stresses.

Conservation of Momentum - Integral form

- Using the Reynolds Transport Theorem as usual, the definition of $\boldsymbol{Q}$ and the expressions in the previous slide, we obtain the following integral form of the conservation of momentum

$$\int_{V_0} \frac{\partial \rho \boldsymbol{u}}{\partial t}\, dV + \int_{S_0} \rho \boldsymbol{u}\boldsymbol{u} \cdot \boldsymbol{n}\, dS = -\int_{S_0} p\boldsymbol{I} \cdot \boldsymbol{n}\, dS + \int_{S_0} \boldsymbol{\tau} \cdot \boldsymbol{n}\, dS + \int_{V_0} \rho \boldsymbol{f}\, dV,$$

Conservation of Momentum - Differential form

- From the integral form of the conservation of momentum

$$\int_{V_0} \frac{\partial \rho \boldsymbol{u}}{\partial t}\, dV + \int_{S_0} \rho \boldsymbol{u}\boldsymbol{u} \cdot \boldsymbol{n}\, dS = -\int_{S_0} p\boldsymbol{I} \cdot \boldsymbol{n}\, dS + \int_{S_0} \boldsymbol{\tau} \cdot \boldsymbol{n}\, dS + \int_{V_0} \rho \boldsymbol{f}\, dV,$$

  if the divergence theorem can be applied (continuous functions, etc), then we can write:

$$\int_{V_0} \left(\frac{\partial \rho \boldsymbol{u}}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}\boldsymbol{u}) \right) dV = \int_{V_0} \left(-\nabla p + \nabla \cdot \boldsymbol{\tau} + \rho \boldsymbol{f} \right) dV,$$

- As we did for the conservation of mass, this can be true for all possible choices of V_0 only if:

$$\frac{\partial (\rho \boldsymbol{u})}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}\boldsymbol{u}) = -\nabla p + \nabla \cdot \boldsymbol{\tau} + \rho \boldsymbol{f}$$

Conservation of Momentum - Differential form

- The last equation in the previous slide:

$$\frac{\partial(\rho \boldsymbol{u})}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}\boldsymbol{u}) = -\nabla p + \nabla \cdot \boldsymbol{\tau} + \rho \boldsymbol{f}$$

- can be written using index notation:

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + \rho f_i$$

- Expanding the first two terms:

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = \rho \frac{\partial u_i}{\partial t} + u_i \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) \right] + \rho u_j \frac{\partial u_i}{\partial x_j} = \rho \frac{Du_i}{Dt}$$

- So we can write:

$$\rho \frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + \rho f_i$$