Fluid Mechanics 4

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Lecture 6

Reynolds transport theorem

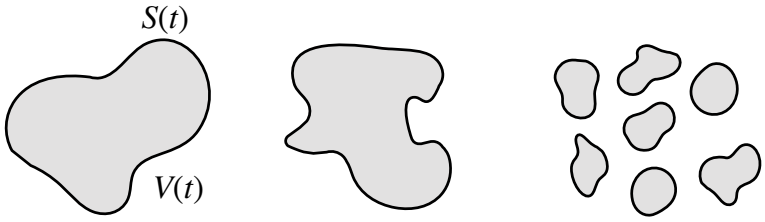


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Equations of motion

- Conservation of Mass
- Conservation of momentum (II Newton Law)
- Conservation of Energy (not covered in this course)

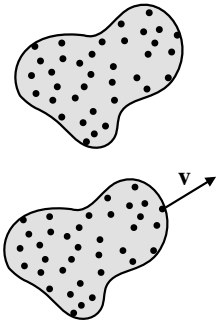
$\Rightarrow$ How are they applied to fluid dynamics?



- The shape depends on time, it changes and can even divide into smaller shapes
- For example, mass is defined as $M(t) = \int_{V(t)} \rho(\boldsymbol{x}, t) dV$
  $\Rightarrow$ In order to solve the equations of motion we need $V(t)$ which depends on the equations of motion.

Definitions

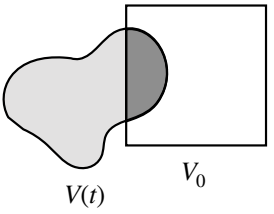
- A **Material Volume** ($V(t)$) is composed by a specific group of fluid particles that move as the fluid flows. This volume is defined by following the fluid particles over time. No particles can enter or exit the volume.
- The volume can change shape, size or connectivity over time; the particles can change state (evaporation) or can react (combustion) but remain inside the Material Volume.
- The mass of the material is constant because it contains the same set of particles
- This approach is often used in the **Lagrangian frame of reference**, where the motion of individual particles is tracked
- The surface $S(t)$ is called a **Material Surface** and moves at the same velocity of the fluid in that point.



Control Volume

Definitions

- A **Control Volume** (V_0) is a fixed (or moving in some cases) region in space (or an observation point) through which fluid flows. Instead of following specific particles, the analysis focuses on how fluid moves into and out of this defined region.
- The boundaries of the control volume can be fixed or moving, but the volume is not tied to individual fluid particles.
- Fluid can flow into and out of the control volume, meaning mass within the control volume can change.
- This approach is used in the **Eulerian frame of reference**, where the properties of the fluid are studied at specific points in space over time.



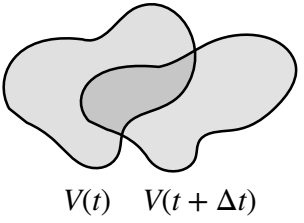
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Extensive and Intensive quantities

Extensive quantity $B(t)$ and Intensive quantity $b(\boldsymbol{x}, t)$

$$B(t) = \int_{V(t)} \rho(\boldsymbol{x}, t) b(\boldsymbol{x}, t) dV \qquad \left[\text{or, in short} \quad B(t) = \int_{V(t)} (\rho b)|_t dV\right]$$

- $B(t)$ depends only on time (mass, energy)
- $b(\boldsymbol{x}, t)$ depends on position and time



$$\frac{dB(t)}{dt} = \frac{d}{dt} \int_{V(t)} \rho(\boldsymbol{x}, t) b(\boldsymbol{x}, t) dV$$

$$\frac{dB(t)}{dt} = \lim_{\Delta t \to 0} \frac{B(t + \Delta t) - B(t)}{\Delta t}$$

with

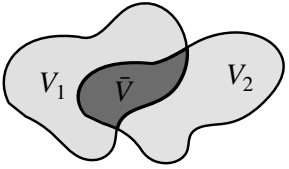
$$B(t + \Delta t) = \int_{V(t+\Delta t)} (\rho b)|_{t+\Delta t} dV$$

$$\frac{dB(t)}{dt} = \lim_{\Delta t \to 0} \frac{\int_{V(t+\Delta t)} (\rho b)|_{t+\Delta t} dV - \int_{V(t)} (\rho b)|_t dV}{\Delta t}$$

Extensive and Intensive quantities

$$\frac{dB(t)}{dt} = \lim_{\Delta t \to 0} \frac{\int_{V(t+\Delta t)} (\rho b)|_{t+\Delta t} dV - \int_{V(t)} (\rho b)|_t dV}{\Delta t}$$

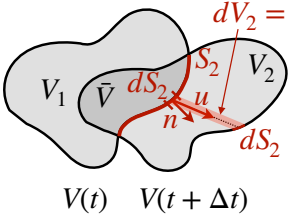
$$V(t) = V_1 + \bar{V}$$
$$V(t+\Delta t) = V_2 + \bar{V}$$

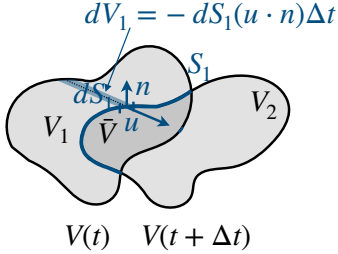


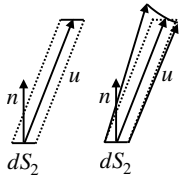
$$\frac{dB(t)}{dt} = \lim_{\Delta t \to 0} \frac{\int_{\bar{V}} (\rho b)|_{t+\Delta t} dV + \int_{V_2} (\rho b)|_{t+\Delta t} dV - \int_{\bar{V}} (\rho b)|_t dV - \int_{V_1} (\rho b)|_t dV}{\Delta t}$$

$$\frac{dB(t)}{dt} = \underbrace{\lim_{\Delta t \to 0} \frac{\int_{\bar{V}} (\rho b)|_{t+\Delta t} - (\rho b)|_t dV}{\Delta t}}_{\int_{V(t)} \frac{\partial (\rho b)}{\partial t} dV} + \lim_{\Delta t \to 0} \frac{\int_{V_2} (\rho b)|_{t+\Delta t} dV}{\Delta t} - \lim_{\Delta t \to 0} \frac{\int_{V_1} (\rho b)|_t dV}{\Delta t}$$

Extensive and Intensive quantities



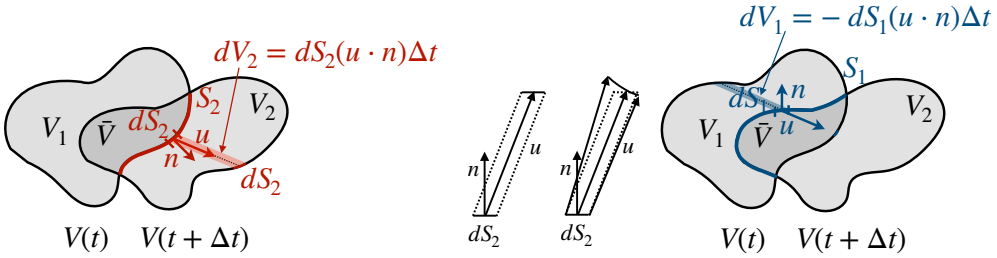




$$\frac{dB(t)}{dt} = \lim_{\Delta t \to 0} \frac{\int_{\bar{V}} (\rho b)|_{t+\Delta t} dV + \int_{V_2} (\rho b)|_{t+\Delta t} dV - \int_{\bar{V}} (\rho b)|_t dV - \int_{V_1} (\rho b)|_t dV}{\Delta t}$$

$$\frac{dB(t)}{dt} = \underbrace{\lim_{\Delta t \to 0} \frac{\int_{\bar{V}} (\rho b)|_{t+\Delta t} - (\rho b)|_t dV}{\Delta t}}_{\int_{V(t)} \frac{\partial (\rho b)}{\partial t} dV} + \underbrace{\lim_{\Delta t \to 0} \frac{\int_{V_2} (\rho b)|_{t+\Delta t} dV}{\Delta t}}_{\lim_{\Delta t \to 0} \int_{S_2} (\rho b)|_{t+\Delta t} \boldsymbol{u} \cdot \boldsymbol{n} dS_2} - \underbrace{\lim_{\Delta t \to 0} \frac{\int_{V_1} (\rho b)|_t dV}{\Delta t}}_{\lim_{\Delta t \to 0} \int_{S_1} (-\rho b)|_{t+\Delta t} \boldsymbol{u} \cdot \boldsymbol{n} dS_1}$$

Extensive and Intensive quantities



$$\frac{dB(t)}{dt} = \lim_{\Delta t \to 0} \frac{\int_{\bar{V}} (\rho b)|_{t+\Delta t} dV + \int_{V_2} (\rho b)|_{t+\Delta t} dV - \int_{\bar{V}} (\rho b)|_t dV - \int_{V_1} (\rho b)|_t dV}{\Delta t}$$

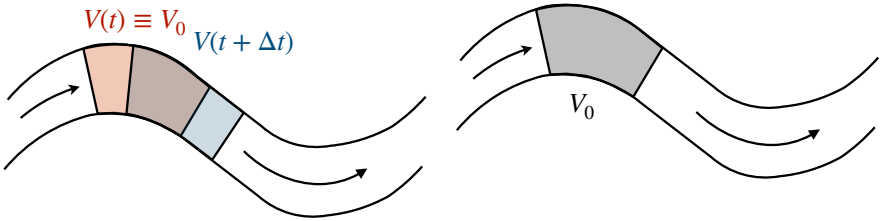
$$\frac{dB(t)}{dt} = \int_{V(t)} \frac{\partial(\rho b)}{\partial t} dV + \lim_{\Delta t \to 0} \int_{S_2} (\rho b)|_{t+\Delta t} \boldsymbol{u} \cdot \boldsymbol{n} dS_2 - \lim_{\Delta t \to 0} \int_{S_1} (-\rho b)|_{t+\Delta t} \boldsymbol{u} \cdot \boldsymbol{n} dS_1$$

$$\frac{dB(t)}{dt} = \int_{V(t)} \frac{\partial(\rho b)}{\partial t} dV + \int_{S(t)} (\rho b) \boldsymbol{u} \cdot \boldsymbol{n} dS$$

Reynolds Transport Theorem $\boxed{\frac{dB(t)}{dt} = \int_{V(t)} \frac{\partial(\rho b)}{\partial t} dV + \int_{S(t)} (\rho b) \boldsymbol{u} \cdot \boldsymbol{n} dS}$

Reynolds Transport Theorem

$$\frac{dB(t)}{dt} = \int_{V(t)} \frac{\partial(\rho b)}{\partial t} dV + \int_{S(t)} (\rho b)\boldsymbol{u} \cdot \boldsymbol{n} dS$$



$$V_0 \equiv V(t)$$
$$S_0 \equiv S_0$$

$$\boxed{\frac{dB(t)}{dt} = \int_{V_0} \frac{\partial(\rho b)}{\partial t} dV + \int_{S_0} (\rho b)\boldsymbol{u} \cdot \boldsymbol{n} dS}$$

- Using the divergence theorem, the Reynolds Transport Theorem

$$\frac{dB(t)}{dt} = \int_{V_0} \frac{\partial(\rho b)}{\partial t} dV + \int_{S_0} (\rho b)\boldsymbol{u} \cdot \boldsymbol{n} dS$$

- can be rewritten as:

$$\frac{dB(t)}{dt} = \int_{V_0} \frac{\partial(\rho b)}{\partial t} dV + \int_{V_0} \nabla \cdot (\rho b \boldsymbol{u}) \, dV$$