Fluid Mechanics 4

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Lecture 4

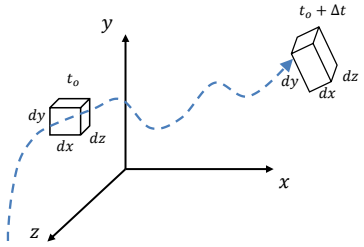
Fluids deformations



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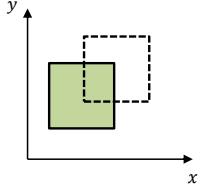
Contributions to the fluid motion

Forces acting on fluid can translate, deform and rotate fluid elements

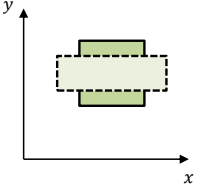


$$\boldsymbol{u}(\boldsymbol{x}) = \underbrace{\boldsymbol{u}_0}_{\text{translation}} + \underbrace{\frac{1}{3}(\nabla \cdot \boldsymbol{u})\boldsymbol{x}}_{\text{linear deformation}} + \underbrace{\frac{1}{2}\boldsymbol{\omega} \times \boldsymbol{x}}_{\text{rotation}} + \underbrace{\boldsymbol{E} \cdot \boldsymbol{x}}_{\text{angular deformation}}$$

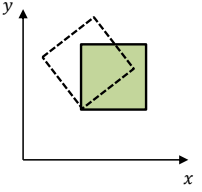
Translation



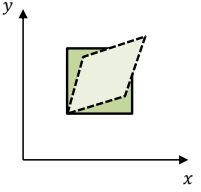
Linear deformation



Rotation



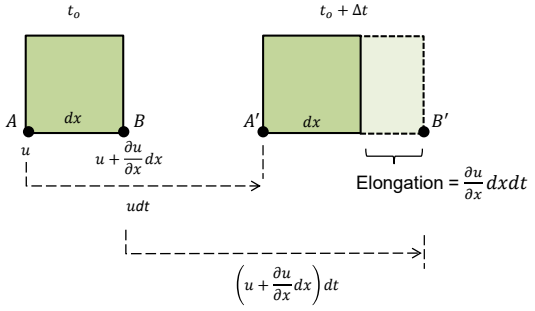
Angular deformation



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Linear Deformation

Consider the x direction with velocity u



- **Strain** $\varepsilon = \Delta L / L$:

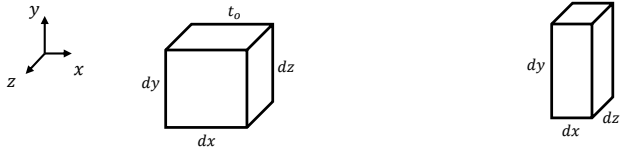
$$d\varepsilon = \left[\left(dx + \frac{\partial u}{\partial x} dx dt\right) - dx\right] / dx = \frac{\partial u}{\partial x} dt$$

- **Strain rate**:

$$\frac{d\varepsilon}{dt} = \dot{\varepsilon} = \frac{\left(\frac{\partial u}{\partial x} dt\right)}{dt} = \frac{\partial u}{\partial x} = S_{xx}$$

Linear Deformation (2)

Consider all three Cartesian coordinate directions x, y and z with u, v and w



- **Overall deformation**

$$d\varepsilon = \frac{\left(dx + \frac{\partial u}{\partial x} dx dt\right) - dx}{dx} + \frac{\left(dy + \frac{\partial v}{\partial y} dy dt\right) - dy}{dy} + \frac{\left(dz + \frac{\partial w}{\partial z} dz dt\right) - dz}{dz} = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) dt$$

- **Deformation rate**:

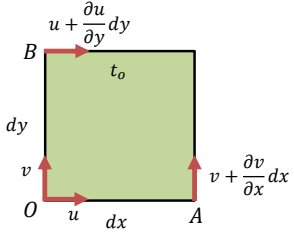
$$\frac{d\varepsilon}{dt} = \dot{\varepsilon} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla \cdot \boldsymbol{u} \Rightarrow \text{Volume dilation rate}$$

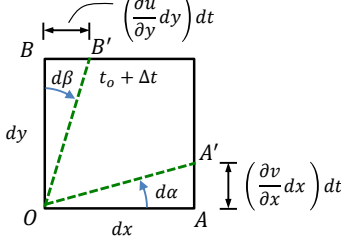
- If incompressible flow:

$$\nabla \cdot \boldsymbol{u} = 0$$

Angular motion and deformation

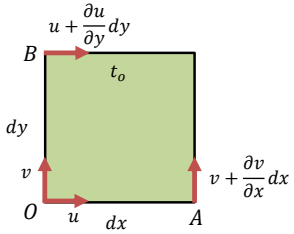
- Take fluid element with segments OA and OB;
- Velocity at O is (u, v)
- Vertical velocity at A is $v + \frac{\partial v}{\partial x} dx$
- Horizontal velocity at B is $u + \frac{\partial u}{\partial y} dy$
- After Δt, OA and OB rotate to $O'A'$ and $O'B'$ through angles $d\alpha$ and $d\beta$

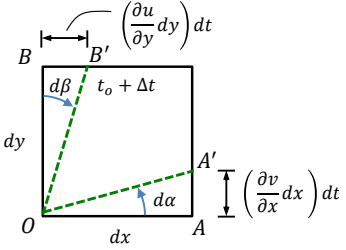




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Angular motion and deformation (2)





- **Angular velocity of** OA: $\hat{\omega}_{OA} = \lim_{t \to 0} \dfrac{d\alpha}{dt}$

$$\underbrace{d\alpha \approx \tan d\alpha}_{\text{small angles}} = \frac{\left(\frac{\partial v}{\partial x} dx dt\right)}{dx} = \frac{\partial v}{\partial x} dt$$

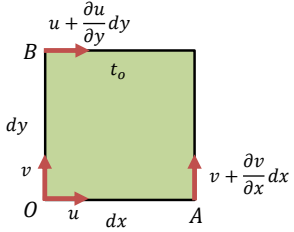
$$\hat{\omega}_{OA} = \lim_{t \to 0} \frac{d\alpha}{dt} = \left[\frac{\left(\frac{\partial v}{\partial x}\right) dt}{dt}\right] = \frac{\partial v}{\partial x}$$

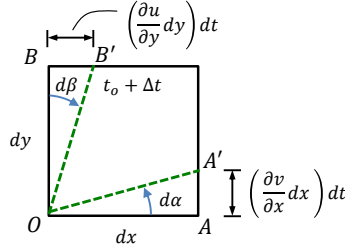
- **Angular velocity of** OB $\hat{\omega}_{OB} = \lim_{t \to 0} \dfrac{d\beta}{dt}$

$$\underbrace{d\beta \approx \tan d\beta}_{\text{small angles}} = \frac{\left(\frac{\partial u}{\partial y} dy dt\right)}{dy} = \frac{\partial u}{\partial y} dt$$

$$\hat{\omega}_{OB} = \lim_{t \to 0} \frac{d\beta}{dt} = \left[\frac{\left(\frac{\partial u}{\partial y}\right) dt}{dt}\right] = \frac{\partial u}{\partial y}$$

Angular rotation





- If $\frac{\partial v}{\partial x}$ is positive, $d\alpha$ is counterclockwise; if $\frac{\partial u}{\partial y}$ is positive, $d\beta$ is clockwise.
- **Right hand rule**: positive rotation is counterclockwise.
- **Rotation** $\hat{\omega}_z$ is defined as the average of $\hat{\omega}_{OA}$ and $\hat{\omega}_{OB}$:
  - $$\hat{\omega}_z = \lim_{t \to 0} \left[\frac{1}{2} \frac{d\alpha - d\beta}{dt} \right] = \frac{1}{2} \left(\hat{\omega}_{OA} - \hat{\omega}_{OB} \right)$$
  - $$\hat{\omega}_z = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

Angular rotation (2)

- Rotation about other axes

$$\hat{\omega}_k = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right)$$

$$\hat{\omega}_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \quad \hat{\omega}_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \quad \hat{\omega}_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

$$\hat{\boldsymbol{\omega}} = \hat{\omega}_x \boldsymbol{e}_1 + \hat{\omega}_y \boldsymbol{e}_2 + \hat{\omega}_z \boldsymbol{e}_3$$

$$\hat{\boldsymbol{\omega}} = \frac{1}{2}\mathrm{curl}\,(\boldsymbol{u}) = \frac{1}{2}\nabla \times \boldsymbol{u} = \frac{1}{2}\begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$\hat{\boldsymbol{\omega}} = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\boldsymbol{e}_1 + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\boldsymbol{e}_2 + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\boldsymbol{e}_3$$

- **Vorticity**: $\boldsymbol{\omega} = 2\hat{\boldsymbol{\omega}} = \nabla \times \boldsymbol{u}$

Angular deformation

Element will rotate around the z-axis as an undeformed element (i.e., $\hat{\omega}_{OA} = -\hat{\omega}_{OB}$) when $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

- Otherwise, rotation will be associated with **Angular Deformation** $d\gamma$
  - $d\gamma = d\alpha + d\beta$
  - $d\gamma > 0$ when the right angle in O decreases

- **Shear strain rate**

$$S_{xy} = \lim_{t\to 0} \frac{d\gamma}{dt} = \lim_{t\to 0} \left[\frac{1}{2} \frac{(\partial v/\partial x)\, dt + (\partial u/\partial y)\, dt}{dt} \right] = \frac{1}{2} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)$$

  - **xy plane**:

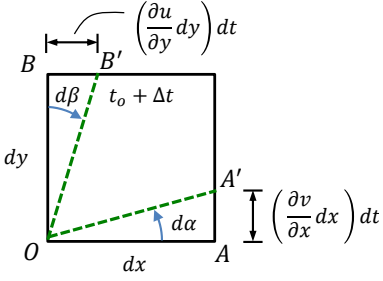
$$S_{xy} = \frac{1}{2} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)$$

  - **yz plane**:

$$S_{yz} = \frac{1}{2} \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right)$$

  - **xz plane**:

$$S_{xz} = \frac{1}{2} \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right)$$



Strain rate tensor

- **Strain rate tensor**

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)$$

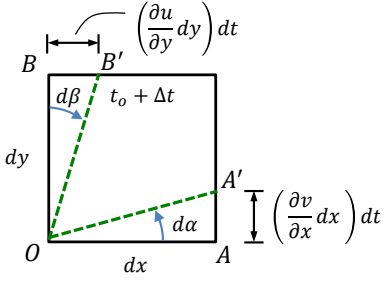
- **Strain rates**
  - **Linear deformation**:

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)$$

  (for $i = j$) (Einstein notation not implied)
  - **Angular deformation**:

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)$$

  (for $i \neq j$)



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Stress-strain rate relationship

- **Simplest relationship**
  1. Linear relationship
  2. Passes through origin
  3. Galilean invariant

- **These criteria are valid for Newtonian fluids**
  - Viscous stresses at every point are linearly related to local strain rate
  - Water, air, fuels, alcohol, glycerine

- **Non-Newtonian fluids**
  - Blood, honey, liquid soap, etc.

