Fluid Mechanics 4

Antonio Attili

School of Engineering, University of Edinburgh

antonio.attili@ed.ac.uk

Lecture 3

Kinematics of Fluids

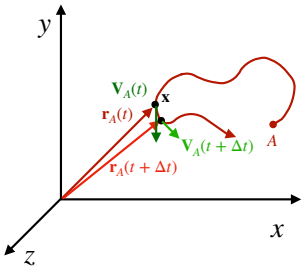


THE UNIVERSITY of EDINBURGH

Kinematics equations of the particle A

$$\boldsymbol{V}_A(t) = \lim_{\Delta t \to 0} \frac{\boldsymbol{r}_A(t + \Delta t) - \boldsymbol{r}_A(t)}{\Delta t}$$

$$\boldsymbol{a}_A(t) = \lim_{\Delta t \to 0} \frac{\boldsymbol{V}_A(t + \Delta t) - \boldsymbol{V}_A(t)}{\Delta t}$$



Where:

- $\boldsymbol{r}_A(t)$ is the position of particle A at time t,
- $\boldsymbol{V}_A(t)$ is the velocity of particle A at time t,
- $\boldsymbol{a}_A(t)$ is the acceleration of particle A at time t.

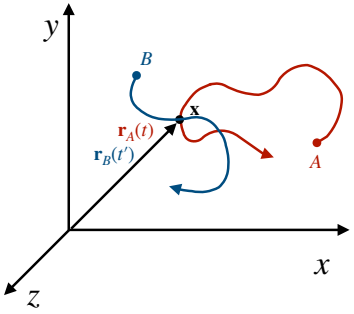
Position, Velocity, and Acceleration Equations

Particle A

$$\boldsymbol{r}_A(t) \equiv \boldsymbol{x}$$
$$\boldsymbol{V}_A(t) = \boldsymbol{u}(\boldsymbol{x}, t)$$
$$\boldsymbol{a}_A(t) = \boldsymbol{a}(\boldsymbol{x}, t)$$

Particle B

$$\boldsymbol{r}_B(t') \equiv \boldsymbol{x}$$
$$\boldsymbol{V}_B(t') = \boldsymbol{u}(\boldsymbol{x}, t')$$
$$\boldsymbol{a}_B(t') = \boldsymbol{a}(\boldsymbol{x}, t')$$



Where:

- The position of particle A at time t is $\boldsymbol{x}$
- So the velocity $\boldsymbol{u}(\boldsymbol{x}, t)$ at point $\boldsymbol{x}$ at time t is $\boldsymbol{V}_A(t)$
- **We define the the velocity in a point $\boldsymbol{x}$ at time t as the velocity of the particle which is in that point at that time**
- Analogously, the position of particle B at time t' is $\boldsymbol{x}$, so the velocity $\boldsymbol{u}(\boldsymbol{x}, t')$ at point $\boldsymbol{x}$ at time t' is $\boldsymbol{V}_B(t)$
- $\boldsymbol{a}_A(t)$ and $\boldsymbol{a}_B(t')$ are the accelerations of particles A and B at times t and t',
- Same for the acceleration

Lagrangian and Eulerian approach

- The Lagrangian approach follows individual fluid particles as they move through space and time,
- The Eulerian approach observes fluid properties at fixed points in space as the fluid moves through those points.

Particle A $\quad \boldsymbol{r}_A(t) \equiv \boldsymbol{x}$

$$\boldsymbol{V}_A(t) = \boldsymbol{u}(\boldsymbol{x}, t)$$

$$\boldsymbol{a}_A(t) = \boldsymbol{a}(\boldsymbol{x}, t)$$

Particle B $\quad \boldsymbol{r}_B(t') \equiv \boldsymbol{x}$

$$\boldsymbol{V}_B(t') = \boldsymbol{u}(\boldsymbol{x}, t')$$

$$\boldsymbol{a}_B(t') = \boldsymbol{a}(\boldsymbol{x}, t')$$

What is the acceleration in $\boldsymbol{x}$ in Eulerian equations?

$$\boldsymbol{a}_A = \frac{d\boldsymbol{V}_A}{dt} = \frac{d\boldsymbol{u}(\boldsymbol{x}, t)}{dt} = \boldsymbol{a}(\boldsymbol{x}, t)$$

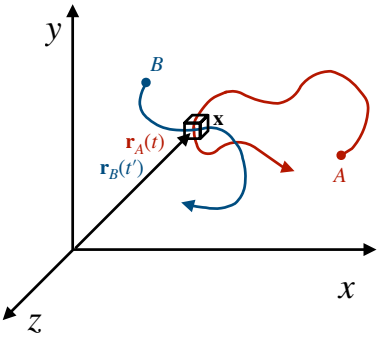
but we know that

$$\boldsymbol{a}(\boldsymbol{x}, t) = \frac{d\boldsymbol{u}(\boldsymbol{x}, t)}{dt} = \frac{\partial \boldsymbol{u}}{\partial t} + \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} \frac{d\boldsymbol{x}}{dt}$$

$$\boldsymbol{a}(\boldsymbol{x}, t) = \frac{\partial \boldsymbol{u}}{\partial t} + \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} \frac{d\boldsymbol{r}_A}{dt} = \frac{\partial \boldsymbol{u}}{\partial t} + \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} \boldsymbol{u}$$

where $\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} \boldsymbol{u}$ is $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$

$$\boxed{\boldsymbol{a}(\boldsymbol{x}, t) = \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = \frac{D\boldsymbol{u}}{Dt}}$$



$\Longrightarrow$ The material derivative

$$\boxed{\frac{D\bullet}{Dt} = \frac{\partial \bullet}{\partial t} + \boldsymbol{u} \cdot \nabla \bullet}$$

Kinematic equations of the fluid

The equation of the acceleration is therefore

$$\boldsymbol{a}(\boldsymbol{x}, t) = \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = \frac{D\boldsymbol{u}}{Dt}$$

which written explicitly is

$$a_x = \underbrace{\frac{\partial u}{\partial t}}_{\text{local acceleration}} + \underbrace{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}}_{\text{convective acceleration}},$$

$$a_y = \underbrace{\frac{\partial v}{\partial t}}_{\text{local acceleration}} + \underbrace{u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}}_{\text{convective acceleration}},$$

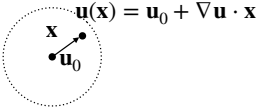
$$a_z = \underbrace{\frac{\partial w}{\partial t}}_{\text{local acceleration}} + \underbrace{u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}}_{\text{convective acceleration}}.$$

From this we can see that

- the equations are non-linear (quadratic) with mixed terms
- all the variables are coupled so they cannot be solved independently $\Rightarrow$ even if I am not interested in what happens along y and z, I still need to solve everything at the same time

Velocity Gradient

$$\nabla \mathbf{u} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix}$$



We can see that we need to know not just what happens in a specific point but we need to know also what happens around that point.

- let's consider a point where the velocity $\boldsymbol{u}_0$ is knows
- let's assume that around that point the velocity exists and is continuous and differentiable
- if we move a small step $\boldsymbol{x}$ the velocity is going to be $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}_0 + \nabla \boldsymbol{u}|_0 \cdot \boldsymbol{x}$

Velocity Gradient

The gradient of $\boldsymbol{u}$ can be divided into different contributions.
We can define the trace of $\nabla \boldsymbol{u}$

$$Tr(\nabla \boldsymbol{u}) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla \cdot \boldsymbol{u}$$

and divide the gradient in two different contributions

$$\nabla \boldsymbol{u} = \frac{1}{3}(\nabla \cdot \boldsymbol{u})\boldsymbol{I} + \boldsymbol{A}$$

where

$$\boldsymbol{A} = \begin{pmatrix} \frac{\partial u}{\partial x} - \frac{\nabla \cdot \boldsymbol{u}}{3} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} - \frac{\nabla \cdot \boldsymbol{u}}{3} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} - \frac{\nabla \cdot \boldsymbol{u}}{3} \end{pmatrix}$$

so the trace of $\boldsymbol{A}$ is $Tr(\boldsymbol{A}) = 0$.

Let's consider now the symmetric and anti-symmetric matrices of $\boldsymbol{A}$:

$$\boldsymbol{E} = sym(\boldsymbol{A}) = \frac{1}{2}(\boldsymbol{A} + \boldsymbol{A}^T)$$

$$\boldsymbol{\Omega} = skw(\boldsymbol{A}) = \frac{1}{2}(\boldsymbol{A} - \boldsymbol{A}^T)$$

$\Rightarrow$ The gradient becomes

$$\nabla \boldsymbol{u} = \frac{1}{3}(\nabla \cdot \boldsymbol{u})\boldsymbol{I} + \boldsymbol{E} + \boldsymbol{\Omega}$$

and

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}_0 + \nabla \boldsymbol{u} \cdot \boldsymbol{x} = \boldsymbol{u}_0 + \frac{1}{3}(\nabla \cdot \boldsymbol{u})\boldsymbol{x} + \boldsymbol{E} \cdot \boldsymbol{x} + \boldsymbol{\Omega} \cdot \boldsymbol{x}$$

Let's define now vorticity $\boldsymbol{\omega}$

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$$

therefore

$$\boldsymbol{\Omega} \cdot \boldsymbol{x} = \frac{1}{2}\boldsymbol{\omega} \times \boldsymbol{x}$$

Vorticity tells us how much a vector field rotates with a solid body rotation.

Finally, the velocity in $\boldsymbol{x}$

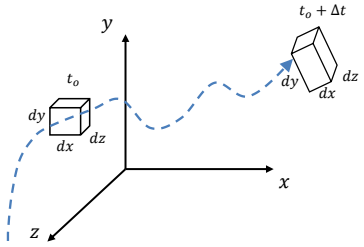
$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}_0 + \frac{1}{3}(\nabla \cdot \boldsymbol{u})\boldsymbol{x} + \boldsymbol{E} \cdot \boldsymbol{x} + \boldsymbol{\Omega} \cdot \boldsymbol{x}$$

becomes

$$\boldsymbol{u}(\boldsymbol{x}) = \underbrace{\boldsymbol{u}_0}_{\text{translation}} + \underbrace{\frac{1}{3}(\nabla \cdot \boldsymbol{u})\boldsymbol{x}}_{\text{compression/dilatation}} + \underbrace{\boldsymbol{E} \cdot \boldsymbol{x}}_{\text{angular deformation}} + \underbrace{\frac{1}{2}\boldsymbol{\omega} \times \boldsymbol{x}}_{\text{rotation}}$$

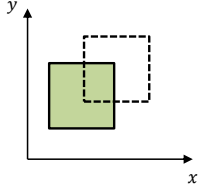
Contributions to the fluid motion

Forces acting on fluid can therefore translate, deform and rotate fluid elements

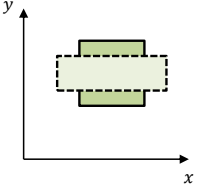


$$\boldsymbol{u}(\boldsymbol{x}) = \underbrace{\boldsymbol{u}_0}_{\text{translation}} + \underbrace{\frac{1}{3}(\nabla \cdot \boldsymbol{u})\boldsymbol{x}}_{\text{linear deformation}} + \underbrace{\frac{1}{2}\boldsymbol{\omega} \times \boldsymbol{x}}_{\text{rotation}} + \underbrace{\boldsymbol{E} \cdot \boldsymbol{x}}_{\text{angular deformation}}$$

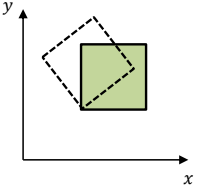
Translation



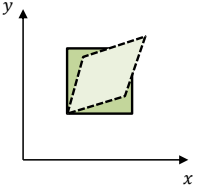
Linear deformation



Rotation



Angular deformation



THE UNIVERSITY of EDINBURGH