Fluid Mechanics 4

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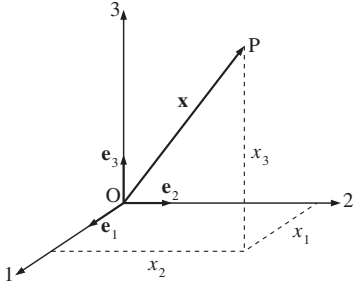
Lecture 2
Math recap



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Physical quantities in fluid mechanics vary in their complexity, and may involve multiple spatial directions:

- **Scalars** (0-th order tensor): single magnitude; e.g., p, T, ρ.
- **Vectors** (1-st order tensor): magnitude & direction; components u_i in a basis $\{\mathbf{e}_i\}$



For example the position vector $\boldsymbol{x}$

$$\boldsymbol{x} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3$$

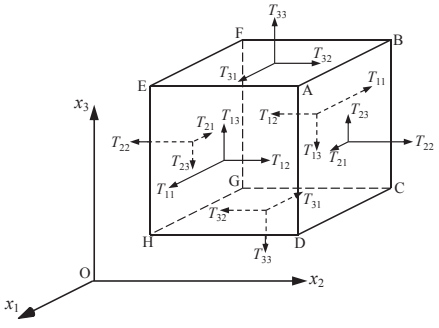
with

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{where} \quad \boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

And the velocity $\boldsymbol{u}$

$$\boldsymbol{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

- **Tensors** (2-nd order tensor): for example the stress tensor T_{ij}; 3×3 components



the first (i) index of T_{ij} denotes the direction of the surface normal, and the second (j) index denotes the force component direction

$$\boldsymbol{T} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

The stresses are positive if they are directed as shown in this figure.

Matrix Inner Product

The product between two 3×3 matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ is $\boldsymbol{P} = \boldsymbol{A} \cdot \boldsymbol{B}$ with

$$P_{ij} = \sum_{k=1}^{3} A_{ik} B_{kj}, \qquad \text{(adjacent index summed)}.$$

so for example

$$\begin{bmatrix} P_{11} & \boxed{P_{12}} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} = \begin{bmatrix} \boxed{A_{11}} & \boxed{A_{12}} & \boxed{A_{13}} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} B_{11} & \boxed{B_{12}} & B_{13} \\ B_{21} & \boxed{B_{22}} & B_{23} \\ B_{31} & \boxed{B_{32}} & B_{33} \end{bmatrix}$$

$$P_{12} = A_{11} B_{12} + A_{12} B_{22} + A_{13} B_{32}$$

Vector Dot and Cross Products

- The **dot product** of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is defined as:

$$\boldsymbol{u} \cdot \boldsymbol{v} = ||\boldsymbol{u}||\ ||\boldsymbol{v}|| \cos(\theta) = u_1 v_1 + u_2 v_2 + u_3 v_3 = \sum_{i=1}^{3} u_i v_i$$

  where θ is the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$.

- The **cross product** of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is defined as:

$$\boldsymbol{u} \times \boldsymbol{v} = (u_2 v_3 - u_3 v_2)\boldsymbol{e}_1 + (u_3 v_1 - u_1 v_3)\boldsymbol{e}_2 + (u_1 v_2 - u_2 v_1)\boldsymbol{e}_3 = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix} = det \begin{bmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$$

Gradient, Divergence and Curl

- The operator ∇ ("del") is defined symbolically by

$$\nabla = \boldsymbol{e}_1 \frac{\partial}{\partial x_1} + \boldsymbol{e}_2 \frac{\partial}{\partial x_2} + \boldsymbol{e}_3 \frac{\partial}{\partial x_3} = \sum_{i=1}^{3} \boldsymbol{e}_i \frac{\partial}{\partial x_i} \equiv \boldsymbol{e}_i \frac{\partial}{\partial x_i}$$

- $\nabla\phi$ is the **gradient** of scalar function ϕ and the result is a vector

$$\nabla\phi = \sum_{i=1}^{3} \boldsymbol{e}_i \frac{\partial \phi}{\partial x_i} \equiv \boldsymbol{e}_i \frac{\partial \phi}{\partial x_i}$$

- $\nabla \cdot \boldsymbol{u}$ is the **divergence** of vector $\boldsymbol{u}$ and the result is a scalar

$$\nabla \cdot \boldsymbol{u} = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} \equiv \frac{\partial u_i}{\partial x_i}$$

- $\nabla \times \boldsymbol{u}$ is the **curl** of $\boldsymbol{u}$ and the result is a vector

$$\nabla \times \boldsymbol{u} = \begin{bmatrix} \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \end{bmatrix} = det \begin{bmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ u_1 & u_2 & u_3 \end{bmatrix}$$

Material Derivative

The **material derivative** (also known as the substantial derivative, convective derivative, or Lagrangian derivative) represents the time rate of change of some quantity (e.g., velocity, temperature) for a specific particle of fluid as it moves through space and time. It takes into account both local changes in the quantity at a point and the changes due to the movement of the fluid.

The material derivative of a scalar quantity ϕ is defined as:

$$\frac{D\phi}{Dt} \equiv \underbrace{\frac{\partial \phi}{\partial t}}_{\text{time or local derivative}} + \underbrace{u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y} + w\frac{\partial \phi}{\partial z}}_{\text{advective derivative}}$$

that can also be written as

$$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \nabla \phi$$

If $\mathbf{v}$ is a vector field (e.g., velocity), the material derivative is applied to each component of the vector:

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{v}$$

NB: $(\boldsymbol{u} \cdot \nabla)$ can be considered an operator

$$(\boldsymbol{u} \cdot \nabla) = u_1\frac{\partial}{\partial x_1} + u_2\frac{\partial}{\partial x_2} + u_3\frac{\partial}{\partial x_3} \text{ so that } (\boldsymbol{u} \cdot \nabla)\boldsymbol{v} = \begin{bmatrix} u_1\frac{\partial v_1}{\partial x_1} + u_2\frac{\partial v_1}{\partial x_2} + u_3\frac{\partial v_1}{\partial x_3} \\ u_1\frac{\partial v_2}{\partial x_1} + u_2\frac{\partial v_2}{\partial x_2} + u_3\frac{\partial v_2}{\partial x_3} \\ u_1\frac{\partial v_3}{\partial x_1} + u_2\frac{\partial v_3}{\partial x_2} + u_3\frac{\partial v_3}{\partial x_3} \end{bmatrix}$$

Material Derivative (2)

The material derivative provides a way to follow a fluid particle and observe how a particular quantity (such as temperature, velocity, or concentration) changes as the particle moves through space and time. It combines the following two effects:

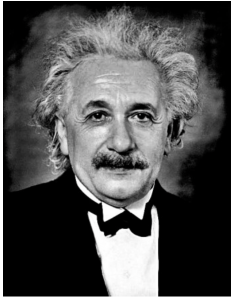
1. The *local* rate of change of the quantity at a fixed point in space, represented by $\frac{\partial \phi}{\partial t}$.
2. The *convective* rate of change, which arises because the fluid particle moves through regions of space where the quantity ϕ might vary. This is represented by $\mathbf{u} \cdot \nabla \phi$, where $\mathbf{u}$ is the velocity of the particle.

The material derivative is crucial in fluid dynamics, especially in the study of how quantities like velocity, temperature, or pressure evolve as fluid particles move, and it plays a central role in the formulation of the Navier-Stokes equations.

Einstein summation convention

Einstein summation convention, also known as the index notation, is a notational convention used in tensor calculus.

- It simplifies expressions involving summation over multiple indices.
- When an index appears twice in a term, it implies summation over that index.



For example let's consider the indices over the set $\{1,2,3\}$ for the three Cartesian coordinates and the function

$$f = \sum_{i=1}^{3} c_i x_i = c_1 x_1 + c_2 x_2 + c_3 x_3$$

In index notation, the expression is

$$f = c_i x_i$$

Here, the repeated index i is summed over, so we do not need to explicitly write the summation symbol.

Einstein summation convention (2)

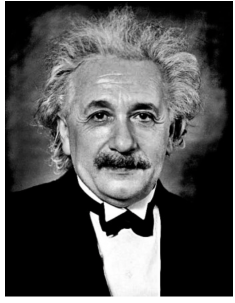
- If we go back to the material derivative

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{v}$$

$$\begin{bmatrix} \frac{Dv_1}{Dt} \\ \frac{Dv_2}{Dt} \\ \frac{Dv_3}{Dt} \end{bmatrix} = \begin{bmatrix} \frac{\partial v_1}{\partial t} \\ \frac{\partial v_2}{\partial t} \\ \frac{\partial v_3}{\partial t} \end{bmatrix} + \begin{bmatrix} u_1 \frac{\partial v_1}{\partial x_1} + u_2 \frac{\partial v_1}{\partial x_2} + u_3 \frac{\partial v_1}{\partial x_3} \\ u_1 \frac{\partial v_2}{\partial x_1} + u_2 \frac{\partial v_2}{\partial x_2} + u_3 \frac{\partial v_2}{\partial x_3} \\ u_1 \frac{\partial v_3}{\partial x_1} + u_2 \frac{\partial v_3}{\partial x_2} + u_3 \frac{\partial v_3}{\partial x_3} \end{bmatrix}$$

  in index form that would be

$$\frac{Dv_j}{Dt} = \frac{\partial v_j}{\partial t} + u_i \frac{\partial v_j}{\partial x_i}$$



- An index that appears once is called a **free index**, in this case j. It means that there are j equations and it must be the same in any term of the equation.
- An index that appears twice is called a **dummy index**, in this case i. It is used for summation. The choice of the dummy symbol is arbitrary and can vary between the different terms.

Contraction of Tensors

- A contraction is an operation that reduces the total order (or rank) of one or more tensors by summing over a repeated index.
- If you have two tensors $\boldsymbol{A}$ and $\boldsymbol{B}$, and you contract over one index, you "pair" one index of $\boldsymbol{A}$ with one index of $\boldsymbol{B}$ and sum over it.

$$C_{ij} = \sum_k A_{ik} B_{kj}$$

Examples:

- **Matrix multiplication:**

$$C_{ij} = A_{ik} B_{kj}$$

- **Dot product of vectors:**

$$u_i v_i = \sum_i u_i v_i$$

- **Trace of a matrix (self-contraction):**

$$\mathrm{tr}(A) = \sum_i A_{ii} = A_{ii}$$

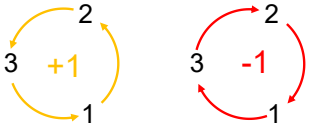
Kronecker Delta and Levi-Civita Symbols

With the **vector cross product** and in particular with the **curl** there are three indices. In order to use the index notation we need to introduce the following symbols.
The Kronecker delta δ_{ij} is defined as:

$$\delta_{ij} \equiv \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad \text{or} \quad \delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The Levi-Civita, or Alternating Tensor, ϵ_{ijk} depends on the order of its argument as follows:

$$\epsilon_{ijk} \equiv \begin{cases} 0 & \text{if any of } i,j,k \text{ are equal} \\ 1 & \text{for positive permutations of } i,j,k \ (1,2,3;\ 2,3,1;\ 3,1,2) \\ -1 & \text{for negative permutations of } i,j,k \ (1,3,2;\ 3,2,1;\ 2,1,3) \end{cases}$$



Vector to vector operations in Index Form

- Cross product

$$\boldsymbol{u} \times \boldsymbol{v} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

in index form

$$\epsilon_{ijk} u_j v_k$$

- Curl

$$\nabla \times \boldsymbol{u} = \begin{bmatrix} \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \end{bmatrix}$$

in index form

$$\epsilon_{ijk} \frac{\partial u_k}{\partial x_j}$$

Vector Calculus Recap

Operator	Symbol	Einstein Convention	Mapping	Example
Gradient of a scalar field ϕ	$\mathrm{grad}(\phi) = \nabla\phi$	$\frac{\partial \phi}{\partial x_i}$	scalar to vector	gradient of pressure field
Divergence of a vector field v	$\mathrm{div}(\mathbf{v}) = \nabla \cdot \mathbf{v}$	$\frac{\partial v_i}{\partial x_i}$	vector to scalar	sinks and sources of mass
Curl of a vector field v	$\mathrm{curl}(\mathbf{v}) = \nabla \times \mathbf{v}$	$\epsilon_{ijk}\frac{\partial v_k}{\partial x_j}$	vector to vector	vorticity of velocity field
Laplacian of a scalar field ϕ	$\Delta\phi = \nabla^2\phi$	$\frac{\partial^2 \phi}{\partial x_i \partial x_i}$	scalar to scalar	diffusion

The Fundamental Theorem of Calculus

- The Fundamental Theorem of Calculus connects differentiation and integration, showing that integration can be reversed by differentiation and vice versa.

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where:

- $f(x)$ is a continuous function.
- $F(x)$ is an antiderivative (or indefinite integral) of $f(x)$, meaning $F'(x) = f(x)$.
- The integral gives the area under the curve of $f(x)$ from a to b.

- The theorem states that the definite integral of a function $f(x)$ over the interval $[a, b]$ is the difference in the values of its antiderivative $F(x)$ at a and b.
- This connects the accumulation of values (integration) with the instantaneous rate of change (differentiation).

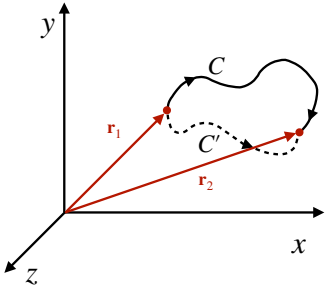
The Gradient Theorem

- The Gradient Theorem (also known as the Fundamental Theorem of Line Integrals) relates the line integral of a gradient field over a curve to the values of a scalar field at the endpoints of the curve.

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}_2) - f(\mathbf{r}_1)$$

where:

- f is a scalar function.
- C is a smooth curve connecting points $\mathbf{r}_1$ and $\mathbf{r}_2$.
- ∇f is the gradient of f.
- $d\mathbf{r}$ is a vector differential along the curve.



- This theorem shows that the line integral of a gradient field depends only on the values of the scalar field at the start and end points of the curve, not on the specific path taken between them.
- As a result, any closed loop integral of a gradient field is zero.

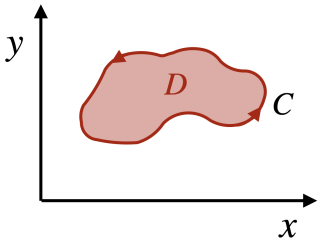
Green's Theorem

- Green's Theorem relates a line integral around a simple, closed curve C to the integral over the plane region D bounded by C.

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA$$

where:

- $\mathbf{F} = F_1(x, y)\boldsymbol{e}_1 + F_2(x, y)\boldsymbol{e}_2$ is a vector field in 2D.
- C is a positively oriented, simple closed curve.
- D is the region enclosed by C.
- $d\mathbf{r}$ is the differential along the curve C.
- dA is the area differential over D.



- Green's Theorem is a fundamental result in vector calculus that connects the circulation of a vector field around a closed curve with the sum of the curl of the field over the region inside the curve.
- It can be thought of as a 2D version of the Fundamental Theorem of Calculus.

Stokes' Theorem

- Stokes' Theorem generalizes Green's Theorem to three dimensions, relating a surface integral of the curl of a vector field over a surface S to a line integral around the boundary of S.

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

where:

- $\mathbf{F}$ is a vector field.
- S is a smooth surface with boundary ∂S.
- $\nabla \times \mathbf{F}$ is the curl of $\mathbf{F}$.
- $d\mathbf{r}$ is the differential vector along the boundary curve ∂S.
- $d\mathbf{S}$ is the differential surface element of S.

- Stokes' Theorem relates the circulation of a vector field along a closed curve (the boundary of a surface) to the curl of the vector field over the surface itself.
- This theorem is a higher-dimensional generalization of both Green's Theorem and the Fundamental Theorem of Calculus.

Gauss's Theorem (Divergence Theorem)

- Gauss's Theorem (also known as the Divergence Theorem) relates the flux of a vector field through a closed surface to the divergence of the field inside the volume enclosed by the surface.

$$\oint_{\partial V} \mathbf{F} \cdot d\mathbf{A} = \iiint_V (\nabla \cdot \mathbf{F})\, dV$$

where:

- $\mathbf{F}$ is a vector field.
- V is a volume in 3D space.
- ∂V is the closed surface (boundary) enclosing V.
- $\nabla \cdot \mathbf{F}$ is the divergence of the vector field $\mathbf{F}$.
- $d\mathbf{A}$ is the differential vector normal to the surface ∂V.
- dV is the volume element inside V.

- The Divergence Theorem states that the total flux of a vector field through the surface ∂V is equal to the integral of the divergence of the field over the volume V enclosed by the surface.
- This theorem connects a local property (divergence) with a global property (flux).