Fluid Mechanics 4

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Lecture 16

Mean-flow Equations. Kinetic energy in turbulent flows.



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Equations of fluid motion

- Continuity equation (mass conservation)

$$\nabla \cdot \mathbf{u} = 0$$

- Navier-Stokes equations (momentum conservation)

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u}$$

- In index notation

$$\frac{\partial u_j}{\partial t} + u_i \frac{\partial u_j}{\partial x_i} = -\frac{1}{\rho} \frac{\partial p}{\partial x_j} + \nu \nabla^2 u_j$$

- In conservative form:

$$\frac{\partial u_j}{\partial t} + \frac{\partial}{\partial x_i}(u_i u_j) = -\frac{1}{\rho} \frac{\partial p}{\partial x_j} + \nu \nabla^2 u_j$$

where the conservative form is obtained observing that:

$$\frac{\partial u_j}{\partial t} + \frac{\partial}{\partial x_i}(u_i u_j) = \frac{\partial u_j}{\partial t} + u_i \frac{\partial u_j}{\partial x_i} + u_j \cancel{\frac{\partial u_i}{\partial x_i}}$$

We can derive evolution equations for all statistical quantities:

- Probability density function:

$$f(v)$$

- Mean:

$$\langle u \rangle = \int_{-\infty}^{\infty} v f(v) dv$$

- Variance:

$$var(u) = \langle u'^2 \rangle = \int_{-\infty}^{\infty} (v - \langle u \rangle)^2 f(v) dv$$

- Why do we care for the computation of statistical quantities?
  - In most applications, we can compute engineering relevant quantities from mean and rms (lift of a wing, thrust of a rocket)
  - Computational cost: solving the mean equation is much cheaper than solving the original one.

Reynolds decomposition and Mean Continuity Equation

- Starting from the Navier-Stokes equations, it is possible to write equations for each statistical object, such as means, PDF, correlation, etc.
- The most basic equation is for the mean (originally obtained by Reynolds in 1884).
- For the derivation of the mean equations, we will use the Reynolds decomposition of the velocity:

$$\boldsymbol{u}(\boldsymbol{x}, t) = \langle \boldsymbol{u}(\boldsymbol{x}, t) \rangle + \boldsymbol{u}'(\boldsymbol{x}, t)$$

- For the continuity equation:

$$\nabla \cdot \boldsymbol{u} = 0$$

- We start by applying the mean:

$$\langle \nabla \cdot \boldsymbol{u} \rangle = 0$$

- since mean and differentiation commute, we obtain very easily the mean continuity equation:

$$\nabla \cdot \langle \boldsymbol{u} \rangle = 0$$

- Also, if we use the Reynolds decomposition in the original continuity equation, we can write

$$\nabla \cdot (\langle \boldsymbol{u} \rangle + \boldsymbol{u}') = 0 \;\Rightarrow\; \nabla \cdot \langle \boldsymbol{u} \rangle + \nabla \cdot \boldsymbol{u}' = 0 \;\Rightarrow\; \nabla \cdot \boldsymbol{u}' = 0$$

Mean Momentum Equation 1

We start from Navier–Stokes in conservative form:

$$\frac{\partial u_j}{\partial t} + \frac{\partial}{\partial x_i}(u_i u_j) = -\frac{1}{\rho}\frac{\partial p}{\partial x_j} + \nu \nabla^2 u_j$$

Apply the mean to each term:

$$\left\langle \frac{\partial u_j}{\partial t} \right\rangle = \frac{\partial \langle u_j \rangle}{\partial t}$$

$$\left\langle \frac{\partial}{\partial x_i}(u_i u_j) \right\rangle = \frac{\partial}{\partial x_i}\langle u_i u_j \rangle$$

$$\left\langle \frac{\partial p}{\partial x_i} \right\rangle = \frac{\partial \langle p \rangle}{\partial x_i}$$

$$\langle \nu \nabla^2 u_j \rangle = \nu \nabla^2 \langle u_j \rangle$$

However, the non-linear term $\langle u_i u_j \rangle$ presents a challenge since it contains a product of velocities, which does not commute with the mean operator.

Mean Momentum Equation 2

Using Reynolds decomposition:

$$\boldsymbol{u}(\boldsymbol{x},t) = \langle \boldsymbol{u}(\boldsymbol{x},t)\rangle + \boldsymbol{u}'(\boldsymbol{x},t)$$

The non-linear term $\langle u_i u_j\rangle$ can be expanded as:

$$\langle u_i u_j\rangle = \langle \left(\langle u_i\rangle + u_i'\right)\left(\langle u_j\rangle + u_j'\right)\rangle = \langle \langle u_i\rangle\langle u_j\rangle + u_i'\langle u_j\rangle + \langle u_i\rangle u_j' + u_i'u_j'\rangle$$

which can be written as:

$$\left\langle \langle u_i\rangle \langle u_j\rangle \right\rangle + \left\langle u_i' \langle u_j\rangle \right\rangle + \left\langle \langle u_i\rangle u_j' \right\rangle + \left\langle u_i' u_j' \right\rangle = \langle u_i\rangle \langle u_j\rangle + \cancel{\langle u_i'\rangle \langle u_j\rangle} + \cancel{\langle u_i\rangle \left\langle u_j' \right\rangle} + \left\langle u_i' u_j' \right\rangle$$

So the averaged Navier–Stokes equations are:

$$\frac{\partial \langle u_j\rangle}{\partial t} + \langle u_i\rangle \frac{\partial \langle u_j\rangle}{\partial x_i} = -\frac{1}{\rho}\frac{\partial \langle p\rangle}{\partial x_i} + \nu \nabla^2 \langle u_j\rangle - \frac{\partial}{\partial x_i}\langle u_i' u_j'\rangle$$

Reynolds stresses (1)

- Reynolds stresses $\langle u_i' u_j' \rangle$ play a crucial role. If $\langle u_i' u_j' \rangle = 0$ then $\boldsymbol{u}$ and $\langle \boldsymbol{u} \rangle$ would have the same equation.
- The Reynolds equations can be written as (general form of momentum conservation)

$$\rho \left[\frac{\partial \langle u_j \rangle}{\partial t} + \langle u_i \rangle \frac{\partial \langle u_j \rangle}{\partial x_i} \right] = \frac{\partial}{\partial x_i} \left[\mu \left(\frac{\partial \langle u_i \rangle}{\partial x_j} + \frac{\partial \langle u_j \rangle}{\partial x_i} \right) - \langle p \rangle \delta_{ij} - \rho \langle u_i' u_j' \rangle \right].$$

- **Viscous stresses:**

$$\mu \left(\frac{\partial \langle u_i \rangle}{\partial x_j} + \frac{\partial \langle u_j \rangle}{\partial x_i} \right)$$

- **Isotropic stresses from the mean pressure field:**

$$-\langle p \rangle \delta_{ij}$$

- **Reynolds stress ("apparent" stress from fluctuations):**

$$-\rho \langle u_i' u_j' \rangle$$

(Usually $\langle u_i' u_j' \rangle$ are called "Reynolds stress", more than the whole term)

Reynolds stresses (2): control-volume view

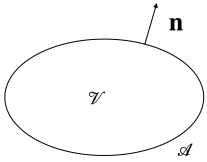
Consider a control volume $\mathcal{V}$ with boundary $\mathcal{A}$ and outward normal $\mathbf{n}$.

- The momentum increase within $\mathcal{V}$ for unit time due to the flow through the boundary $\mathcal{A}$ (momentum per unit volume $\rho\boldsymbol{u}$, flow rate per unit area $-\boldsymbol{u}\cdot\boldsymbol{n}$)

$$\dot{\mathbf{M}} = \int_{\mathcal{A}} \rho\,\mathbf{u}\,(-\mathbf{u}\cdot\mathbf{n})\,dA$$

For each component of $\dot{\boldsymbol{M}}$, the mean is:

$$\begin{aligned}\langle \dot{M}_j \rangle &= \int_{\mathcal{A}} -\rho\left(\langle u_i\rangle\langle u_j\rangle + \langle u_i' u_j'\rangle\right) n_i\, dA \\ &= \int_{\mathcal{V}} -\rho\,\frac{\partial}{\partial x_i}\left(\langle u_i\rangle\langle u_j\rangle + \langle u_i' u_j'\rangle\right) dV,\end{aligned}$$



using the divergence theorem.

Therefore, for the control volume $\mathcal{V}$, the Reynolds stress $-\rho\dfrac{\partial\langle u_i' u_j'\rangle}{\partial x_i}$, originates from the mean momentum flux due to the fluctuating velocity on the boundary $\mathcal{A}$: $-\rho\,\langle u_i' u_j'\rangle\, n_i$

Properties of the Reynolds stress tensor

The Reynolds stresses are the component of a symmetric second order tensor.

- Diagonal components $\langle u_i' u_i' \rangle$ are normal stresses
- Off-diagonal components $\langle u_i' u_j' \rangle$ with $(i \neq j)$ are shear stresses.
- **Turbulent kinetic energy (TKE)** is defined as:

$$k = \frac{1}{2} \langle \mathbf{u}' \cdot \mathbf{u}' \rangle = \frac{1}{2} \langle u_i' u_i' \rangle$$

- Normal and shear stress definition depends on the coordinate system. Then we introduce **isotropic** and **anisotropic** (deviatoric) stresses

$$\text{Isotropic: } \frac{2}{3} k \delta_{ij} = \frac{1}{3} \langle u_i' u_i' \rangle \delta_{ij}$$

$$\text{Anisotropic: } a_{ij} = \langle u_i' u_j' \rangle - \frac{2}{3} k \, \delta_{ij}.$$

- Only the anisotropic components are effective in transporting momentum:

$$\rho \frac{\partial \langle u_i' u_j' \rangle}{\partial x_i} + \frac{\partial \langle p \rangle}{\partial x_j} = \rho \frac{\partial a_{ij}}{\partial x_i} + \frac{\partial}{\partial x_j} \left(\langle p \rangle + \frac{2}{3} k \, \delta_{ij} \right).$$

The isotropic part can be absorbed into a *modified* pressure: $\langle p \rangle + \frac{2}{3} k \, \delta_{ij}$

Mean continuity equation

$$\nabla \cdot \langle \mathbf{u} \rangle = 0.$$

Mean momentum equation

$$\frac{\partial \langle u_j \rangle}{\partial t} + \langle \mathbf{u} \rangle \cdot \nabla \langle u_j \rangle = \nu \nabla^2 \langle u_j \rangle - \frac{\partial \langle u_i' u_j' \rangle}{\partial x_i} - \frac{1}{\rho} \frac{\partial \langle p \rangle}{\partial x_j}.$$

- Unknowns (4 independent variables): $\langle u_1 \rangle, \langle u_2 \rangle, \langle u_3 \rangle, \langle p \rangle$
- Additional unknowns: Reynold stresses $\langle u_i' u_j' \rangle$ (6 distinct components)
- $\Rightarrow$ Unclosed set of equations
- Models needed to **close the equations**

Turbulent viscosity hypothesis

- The **turbulent viscosity hypothesis** (Boussinesq, 1877) is mathematically analogous to the constitutive law of a Newtonian fluid. **Reynolds stresses are proportional to the rate of strain**

$$-\rho\, a_{ij} \equiv -\rho \left(\langle u_i' u_j' \rangle - \frac{2}{3} k\, \delta_{ij} \right) = \rho\, \nu_T(\boldsymbol{x}, t) \left(\frac{\partial \langle u_i \rangle}{\partial x_j} + \frac{\partial \langle u_j \rangle}{\partial x_i} \right).$$

  where $\nu_T(\boldsymbol{x}, t)$ is the turbulent (eddy) viscosity.

- Then the mean momentum with the turbulent viscosity hypothesis becomes

$$\frac{\partial \langle u_j \rangle}{\partial t} + \langle \mathbf{u} \rangle \cdot \nabla \langle u_j \rangle = \frac{\partial}{\partial x_i} \left[\nu_{\text{eff}} \left(\frac{\partial \langle u_i \rangle}{\partial x_j} + \frac{\partial \langle u_j \rangle}{\partial x_i} \right) \right] - \frac{1}{\rho} \frac{\partial}{\partial x_j} \left(\langle p \rangle + \frac{2}{3} \rho k \right),$$

  where $\nu_{\text{eff}}(\boldsymbol{x}, t) = \nu + \nu_T(\boldsymbol{x}, t)$ being an *effective* viscosity (molecular plus turbulent), and $k = \langle u_i u_i \rangle$ being the turbulent kinetic energy

- **Done!**
- **Except, we don't know ν_T $\Rightarrow$ we require a model!**

Prandtl's mixing-length model

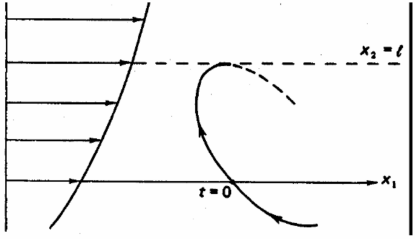
- A model for the eddy viscosity. From dimensional analysis: $\nu_T \sim \ell^* u^*$
- **Assumption:** Turbulent transport due to eddy with length scale ℓ_m (mixing length) and velocity fluctuation u^*

- The velocity fluctuation u^* is approximated from the following simplified view
- The turbulent eddy will transport fluid from x_1(region with low momentum $\rho\langle u_1\rangle$) to x_2 (region with high momentum $\rho\langle u_2\rangle$) over a distance ℓ_m creating fluctuations ρu^*
- We approximate the velocity fluctuation $\rho u^* \sim \rho\langle u_2\rangle - \rho\langle u_1\rangle$
- Using the Taylor expansion

$$\langle u_2\rangle = \langle u_1\rangle + \frac{\partial\langle u\rangle}{\partial x}\ell_m \text{ and } u^* = \left|\frac{\partial\langle u\rangle}{\partial x}\right|\ell_m$$

then

$$\nu_T = \ell_m^2 \left|\frac{\partial\langle u\rangle}{\partial x}\right|.$$



- ℓ_m **must be specified.**
  This can be done for well-understood specific flows (e.g., boundary layers over wings)

Some observations on this simple model

- Specification of $\nu_T(\mathbf{x}, t)$ solves the closure problem and the mean flow equations can be solved.
- At high Reynolds number (away from walls), ν_T scales with a velocity scale $\mathcal{U}$ and length scale $\mathcal{L}$ of the flow.
- Thus ν_T/ν increases and molecular transport becomes negligible.

Kinetic energy in turbulent flows

- Kinetic energy per unit mass:

$$E(\mathbf{x}, t) = \frac{1}{2}\,\mathbf{u} \cdot \mathbf{u}.$$

- Using the Reynolds decomposition, it can be written as:

$$E = \frac{1}{2}\big(\langle \mathbf{u} \rangle \cdot \langle \mathbf{u} \rangle + \langle \mathbf{u}' \cdot \mathbf{u}' \rangle\big)$$

- We observe that the kinetic energy can be decomposed into two parts

$$E = \underbrace{\frac{1}{2}\langle \mathbf{u} \rangle \cdot \langle \mathbf{u} \rangle}_{\tilde{E}} + \underbrace{\frac{1}{2}\langle \mathbf{u}' \cdot \mathbf{u}' \rangle}_{k}.$$

  - $\tilde{E}$: mean-flow kinetic energy;
  - k: turbulent kinetic energy.

- The turbulent kinetic energy is one of the most important quantity in turbulence theory and modelling

From the Navier-Stokes equations and the Reynolds decomposition, it is possible to derive the equations for:

Mean kinetic energy

$$\frac{\partial \tilde{E}}{\partial t} + \langle \mathbf{u} \rangle \cdot \nabla \tilde{E} + \nabla \cdot \langle \mathbf{T} \rangle = -\bar{\varepsilon} - \mathcal{P}$$

with

$$\langle T_i \rangle = \langle u_j \rangle \langle u_i' u_j' \rangle + \langle u_i \rangle \frac{\langle p \rangle}{\rho} - 2\nu \langle u_j \rangle \langle S_{ij} \rangle$$

$$\bar{\varepsilon} = 2\nu \langle S_{ij} \rangle \langle S_{ij} \rangle \qquad \langle S_{ij} \rangle = \frac{1}{2}\left(\frac{\partial \langle u_i \rangle}{\partial x_j} + \frac{\partial \langle u_j \rangle}{\partial x_i} \right)$$

Turbulent kinetic energy

$$\frac{\partial k}{\partial t} + \langle \mathbf{u} \rangle \cdot \nabla k + \nabla \cdot \mathbf{T}' = \mathcal{P} - \varepsilon$$

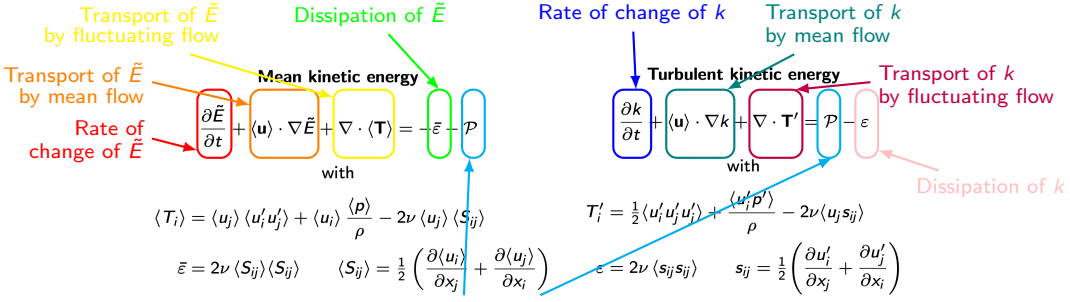
with

$$T_i' = \frac{1}{2}\langle u_i' u_j' u_j' \rangle + \frac{\langle u_i' p' \rangle}{\rho} - 2\nu \langle u_j s_{ij} \rangle$$

$$\varepsilon = 2\nu \langle s_{ij} s_{ij} \rangle \qquad s_{ij} = \frac{1}{2}\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i} \right)$$

Kinetic energy equations

From the Navier-Stokes equations and the Reynolds decomposition, it is possible to derive the equations for:



Production of turbulent kinetic energy k: $\mathcal{P} = -\left\langle u_i' u_j' \right\rangle \frac{\partial \langle u_i \rangle}{\partial x_j}$

- $\tilde{E}$ and k are transported (convected) by the mean and fluctuating velocity fields
- $\tilde{E}$ is dissipated due to the presence of mean velocity gradients S_{ij} (weak, often negligible)
- k is dissipated due to the presence of fluctuating gradients s_{ij} (strong)
- $\mathcal{P}$ (> 0) **moves energy from mean to fluctuations** (Sink in the equation of $\tilde{E}$, Source in the equation of k)

Take-away messages

- The equations governing mean fields are not closed (the number of unknowns is larger than the number of equations)
- Specification of $\nu_T(\mathbf{x}, t)$ (in the spirit of the turbulent viscosity hypotheses) solves the closure problem and the mean flow equations can be solved.
- At high Reynolds number (and far from walls) ν_T scales with the characteristic velocity $\mathcal{U}$ and length $\mathcal{L}$ of the flow. So ν_T/ν increases and molecular transport becomes negligible.
- In the absence of a forcing term and additional phenomena that inject energy into the system (magnetism, gravity) turbulent fluctuations are generated only by the action of a non-homogeneous mean fields (shear) on the Reynolds stresses. **Transfer of energy from mean to fluctuations.**

$$\mathcal{P} = -\langle u_i' u_j' \rangle \frac{\partial \langle u_i \rangle}{\partial x_j}$$