Fluid Mechanics 4

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Lecture 15

Introduction to turbulent flows. Statistical description of random phenomena.



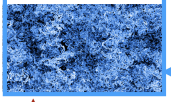
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Roadmap

First principles:
$F = ma$
Mass conservation
Constitutive relations

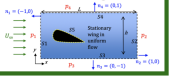
NAVIER STOKES EQUATIONS (+CONTINUITY)
⇒impossible to solve analytically

TURBULENCE
Turbulence theory
Law of the Wall



If $Re \uparrow$ the solution becomes unstable (turbulent)

Global quantities (lift, drag)



Integral equations

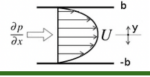
Hypotheses and observation to simplify the equations

Bernoulli relation between two points in the field
$p_1 + \frac{1}{2}\rho V_1^2 + \rho g h_1 = p_2 + \frac{1}{2}\rho V_2^2 + \rho g h_2$

Steady, inviscid, incompressible…

Simple geometries

Analytical solutions of NS ⇒ solution of the velocity field



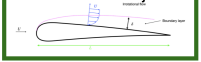
$\mu = 0$
…

Ideal flows
Potential flows
⇒ solution of the velocity field



Close to a wall or object $\mu \neq 0$

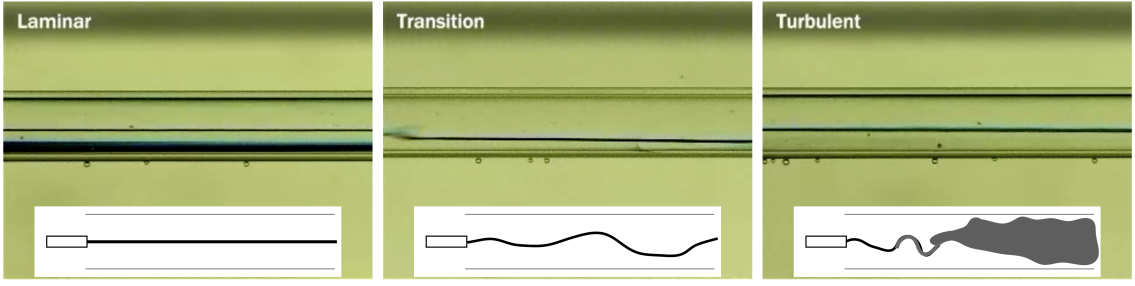
Boundary layer theory close to the object + Potential flow far from the object



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Reynolds experiment

- In **1883** Osborne Reynolds performed a classical experiment on fluid flows.
- He noted that:
  - At low velocities, the layer of dyed water kept its shape
  - At higher velocities, the dyed layer **broke apart** and **diffused** through the rest of the water



Original paper by O. Reynolds: `https://royalsocietypublishing.org/doi/10.1098/rspl.1883.0018`

Turbulent flows

Turbulence is the most common state of fluid flow.

It can be easily observed:

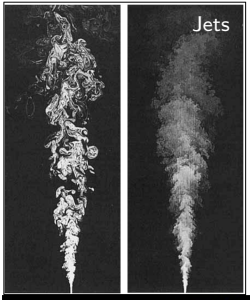
- Water in a river
- Smoke from a chimney
- Clouds
- Rocket plume
- Heart beats are due to turbulent blood flow


Cloud


Plume

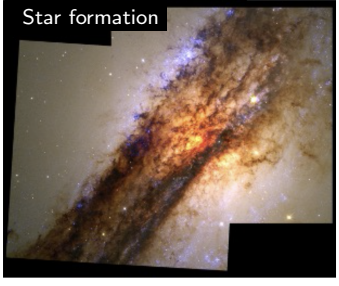

Shinmoedake eruption (2011)

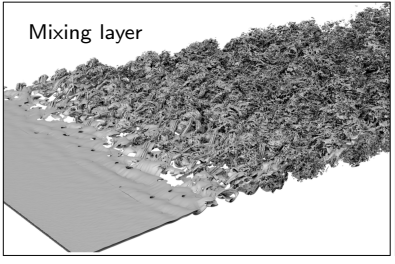
Examples of Turbulent Flows

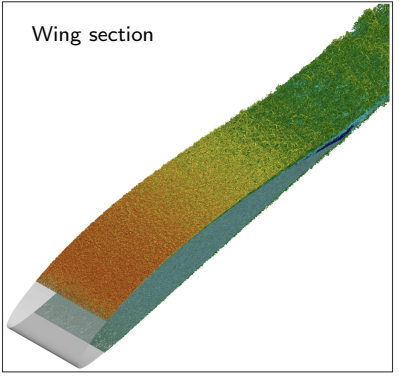






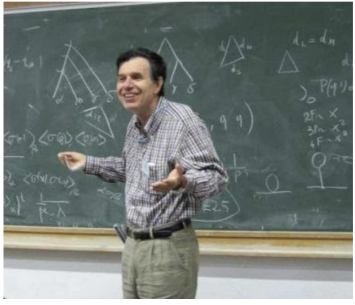






Turbulence is an open scientific problem

- **"The last unsolved problem of classical physics"** (Nobel laureate Richard Feynman).
- "I am an old man; when I die and go to heaven, I'll ask two things:
  Why quantum mechanics? Why turbulence?
  And about the former I am rather optimistic."
  (A quote by Sir Horace Lamb or maybe Werner Heisenberg or Albert Einstein)



2021 Nobel Prize in Physics: Giorgio Parisi

"for the discovery of the interplay of disorder and fluctuations in physical systems from atomic to planetary scales"

J. Phys. A: Math. Gen. **17** (1984) 3521-3531. Printed in Great Britain

On the multifractal nature of fully developed turbulence and chaotic systems

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Received 1 June 1984

Velocity transformation for compressible wall-bounded turbulent flows with and without heat transfer

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Contributed by Parviz Moin, July 14, 2021 (sent for review June 16, 2021; reviewed by Sergio Pirozzoli and Neil Sandham)

In this work, a transformation, which maps the mean velocity profiles of compressible wall-bounded turbulent flows to the incompressible velocity profile, is proposed. The proposed transformation successfully collapses adiabatic and cold-wall, supersonic and hypersonic, zero- and adverse-pressure-gradient turbulent boundary layers with a wide range of friction Reynolds numbers. The proposed transformation successfully collapses adiabatic supersonic turbulent channel flows with friction Reynolds numbers ranging from 250 to 2,000 and a cold-wall channel flow with a friction Reynolds number of 1,000. The transformation is also applied to a hypersonic cold-wall turbulent boundary layer with an edge Mach number of 10 and a friction Reynolds number of 2,540. The proposed transformation is based on physical arguments, and we show that such a general transformation exists for compressible wall-bounded turbulence regardless of the wall thermal condition.

turbulence | wall modeling | law of the wall | mean velocity profile | Mach invariance

It is well known that incompressible wall-bounded turbulent flows at high Reynolds numbers exhibit nearly universal mean streamwise velocity profiles versus the wall-normal coordinate. This is the basis for the law of the wall, which states that, with respect to the friction velocity and the kinematic viscosity, these near, compressible flows with nonnegligible fluctuation Mach numbers, and (2) flows with heat transfer at the wall. Assumption (1) hypothesizes that, as long as the turbulent Mach number (based on the root-mean-square fluctuating velocity) is sufficiently small, compressibility effects are small and the flow dynamics like incompressible counterparts by accounting for the variation in mean properties (density, viscosity, etc.). Such an assumption is justified for flows at low Mach numbers and has also great practical importance for reduced-order turbulence modeling, since it would allow established incompressible RANS models to be used for compressible flows. Since the pioneering work of Van Driest (2), many variants have been proposed over the past decades (e.g., refs. 3–6), but, as noted by Smits and Dussauge (7), despite much effort, no unifier exists in canonical compressible wall-bounded flows. In this work, the strengths and limitations of existing transformations are discussed, and a new transformation with better performance is proposed, validated, and compared with existing approaches for a wide range of compressible wall-bounded turbulent flows.

The first successful velocity transformation is developed by Van Driest (2) by assuming that compressible wall-bounded turbulent flows have a constant turbulent shear stress layer sandwiched between the viscous sublayer and the outer layer. This implies that the nondimensional mean shear, $S_v^+ \equiv \left(1+\overline{\rho'u'}/\overline{\rho}\overline{u}'\overline{v}'\right)\partial\overline{u}/\partial y^+ = \left(\sqrt{\overline{\rho}/\overline{\rho}_w}+\overline{\rho'u'}/\overline{\rho}_w\overline{u}'\overline{v}'\right)\partial\overline{u}/\partial y^+$, is a Mach-number-independent constant of unity, where $\overline{\rho}_w$ is the wall density and $\overline{\rho'u'}/\overline{\rho}_w\overline{u}'\overline{v}'$ indicates a transformation/scaling by the Favre velocity $u_v = u\sqrt{\overline{\rho}/\overline{\rho}_w}$, the viscous length scale $\delta_v = \nu_w/u_\tau$ [1 in p.1], and ρ_w, where $\overline{\rho}_w$, ρ_w, and μ_w are the shear stress, density, and viscosity at the wall, respectively. Integrating S_v^+ with the definition, $U^+ = U/u_\tau = \int_0^{y/\delta_v} S_v^+ dy^+ = \int_0^{y/\delta_v} \sqrt{\overline{\rho}/\overline{\rho}_w} dy^+$.

If a particular nondimensionalization of the mean shear profile is unity, then the mean velocity profile can be found by integration to recover the law of the wall, that formula can be referred to as a successful compressible velocity transformation. The van Driest transformation has been successfully achieved in adiabatic boundary layer flows, since the transformed velocity profiles collapse well to incompressible reference data (e.g., refs. 8 and 9). However, for flows with cold walls (i.e., wall-bounded layers and channel flows, the transformation fails, and the incompressible law of the wall is not recovered (8, 9 [1]).

Fig. 1. Schematic of the Van Driest transformation and its underlying assumption with the mean general proposition of turbulent equilibrium. Specifically, $P(\phi,\psi)$, where P and ϕ denote the probability density function and the sample space [2], is a statement that the turbulence structure is in equilibrium across the entire inner layer of the boundary layer. Further, the van Driest transformation assumes that the turbulent shear stress $\overline{\rho'u'}/\overline{\rho}_w\overline{u}'\overline{v}'/\overline{\rho}_w$ (3) and $\kappa^+ \equiv \kappa/\delta_v$ are Mach-independent

Significance

Near a solid wall, the mean velocity of an incompressible flow is a nearly universal function of the distance from the wall when appropriately transformed (nondimensionalized). This relationship, known as the law of the wall, is a cornerstone of turbulence and velocity. It has not been established how to transform a dimensional velocity profile into a universal one for compressible turbulent flows with heat transfer and more broadly applicable than existing approaches. This improvement results from deploying different physical arguments than in the previous transformations. Moreover, the transformation can be used to extend incompressible turbulence models to compressible turbulent flows, such as those for aerospace vehicles for high-speed transportation and planetary entry.

Author contributions: K.P.G. and L.F. designed the research; K.P.G. performed the research; K.P.G. and P.M. contributed new supercomputer tools; K.P.G. and L.F. analyzed data; K.P.G. and L.F. wrote the paper; and P.M. advised on the research and reviewed and edited the paper.

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The authors declare no competing interest.

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ANNUAL REVIEWS

Annual Review of Fluid Mechanics

Turbulence Modeling in the Age of Data

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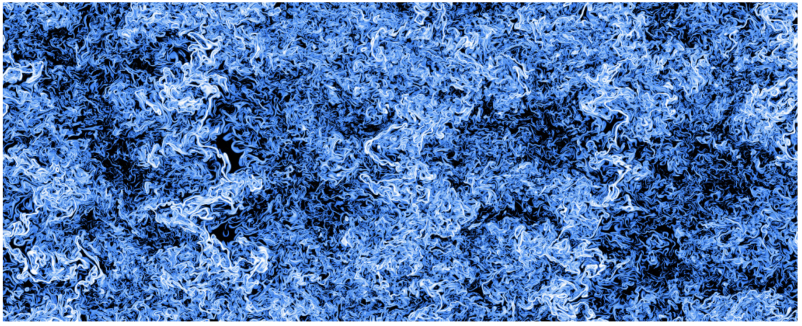
First published as a Review in Advance on September 19, 2018

Keywords

turbulence modeling, statistical inference, machine learning, data-driven modeling, uncertainty quantification

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What is turbulence?

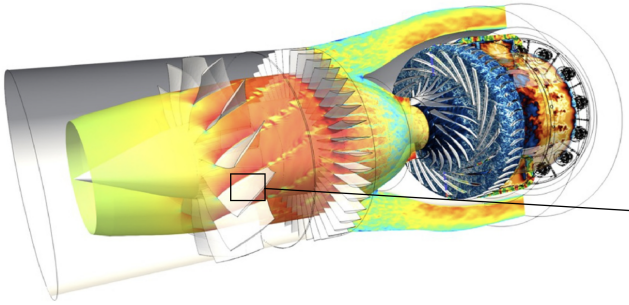


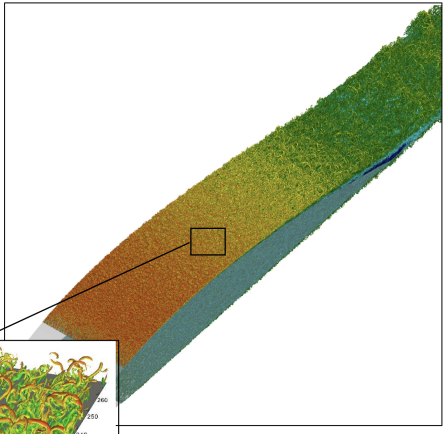
Giving a precise definition of turbulence is difficult. We can list some of its characteristics:

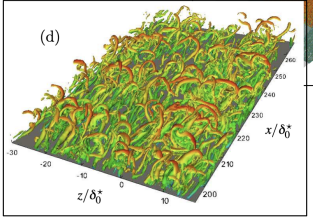
- **Irregularity or randomness.** We must rely on statistics.
- **Three-dimensionality.** Even if two-dimensional turbulent flows exist (e.g., some meteorological flows in the atmosphere), turbulence in unconfined and non-rotating configurations is 3D
- **Diffusivity.** Which causes rapid mixing and increased rate of momentum, heat, and mass transport.
- **Rotational.** Turbulence is characterised by intense vorticity fluctuations.
- **Dissipation.** Turbulent flows are highly dissipative and decay quickly if energy is not injected into the system.

High Reynolds number: $Re = \frac{UL}{\nu}$

Turbulence and scales





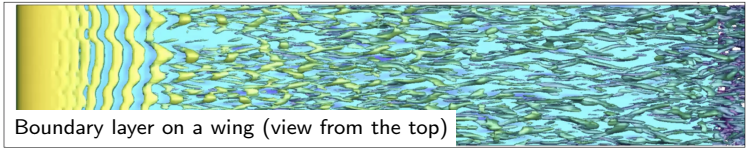


Huge range of scales

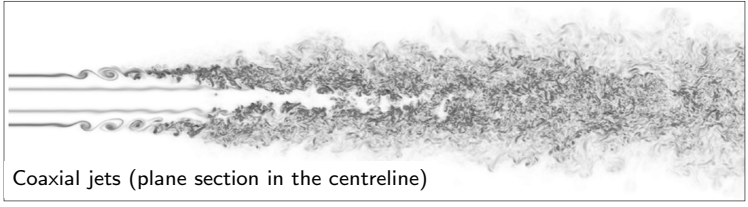
- *Large scales:* size of the system
- *Small scales:*
  - **Small vortices (eddies):** The smallest lengthscale (Kolmogorov scale) decreases as $Re^{-3/4}$
  - **Boundary layer:** The characteristic thickness of the boundary layer decreases as $Re^{-0.85}$

Transition to turbulence

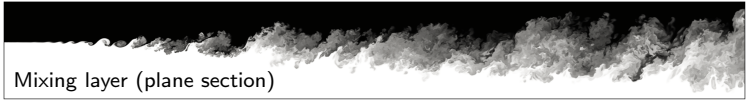
- Turbulence often originates from transition
- A laminar flow becomes unstable and three-dimensional
- Unsteady, chaotic, random features develop in the flow



Boundary layer on a wing (view from the top)

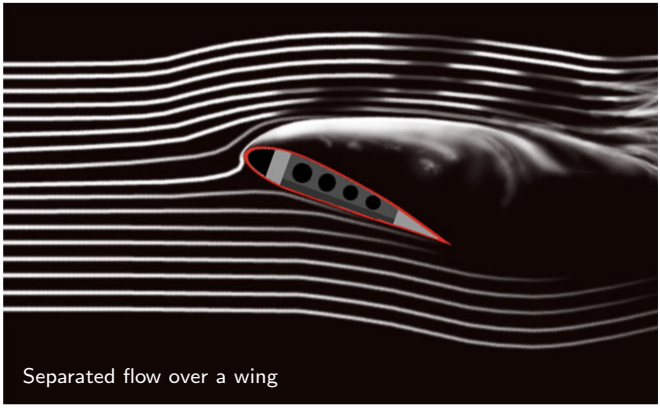


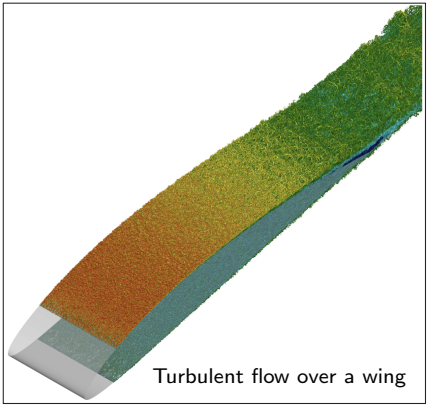
Coaxial jets (plane section in the centreline)



Mixing layer (plane section)

Transition vs Separation





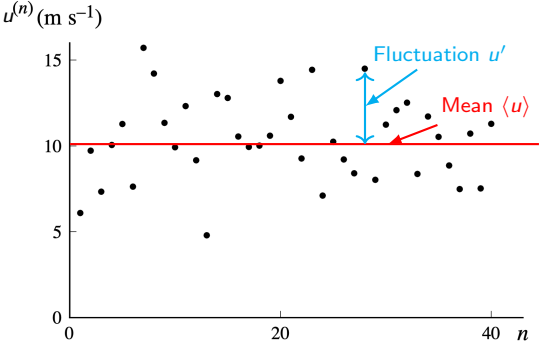
Flow separation and flow transition are two different phenomena:

- **Separation:** Flow streamlines cannot follow the body geometry, usually due to curvature and/or adverse pressure gradients.
- **Laminar-to-turbulence transition:** If the Reynolds number is high enough, a laminar flow can become unstable and transition to turbulence.

Mean and fluctuations

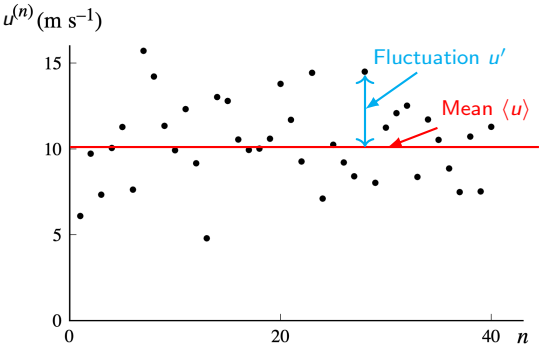
Consider a turbulent flow experiment that is repeated a number of times (under the same conditions) and the velocity $u(\boldsymbol{x})$ is measured in the point $\boldsymbol{x}$ for each repetition n





- **Turbulence is random**: The velocity is different for each repetition of the experiment.
- We need to use statistical quantities for turbulent flows.
- Reynolds decomposition
  velocity = mean + fluctuations: $u = \langle u \rangle + u'$

The random nature of turbulence



- **Observation:** The velocity is different for each repetition of the experiment
- **Question:** Navier-Stokes equations are deterministic; then, why are the solutions random?

- **This is caused by two important aspects:**
  1. In any flow there are, unavoidably, perturbations in the initial conditions, boundary conditions, etc.
  2. Turbulence has an extreme sensitivity to these perturbations.

We tried to specify the same set of conditions for each repetition of the experiment, but this is actually impossible.

The temperature of the fluid, the vibrations of the apparatus, etc., can be accurately controlled, but minor variations will never be completely eliminated.

Lorenz equations and deterministic chaos

Time dependent system with 3 state variables:

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= \rho x - y - xz \\ \dot{z} &= -\beta z + xy \end{aligned}$$

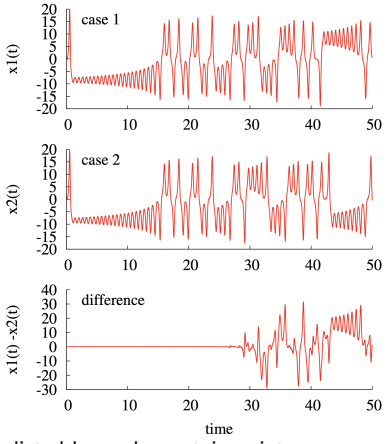
This system was first obtained for free convection in a horizontal liquid layer. The dynamic system also describes some other processes:

- Convection in a toroidal pipe
- Single-mode laser

Assuming $\sigma = 10$; $\beta = 8/3$; $\rho = 28$ for two cases with slighly different initial conditions $[x(0), y(0), z(0)]$:

- case 1: $[x(0), y(0), z(0)] = [0.1, 0.1, 0.1]$
- case 2: $[x(0), y(0), z(0)] = [0.100001, 0.1, 0.1]$

$x_1(t)$ and$x_2(t)$ are initially the same, but by $t \approx 30$ they become very different.

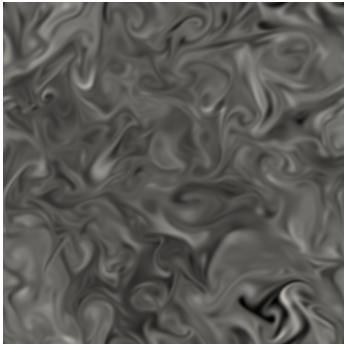


Extreme sensitivity to initial conditions. The system state cannot be predicted beyond a certain point:

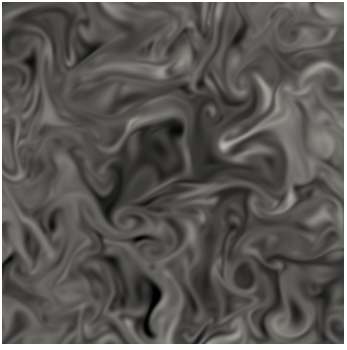
Deterministic chaos

The random nature of Turbulence

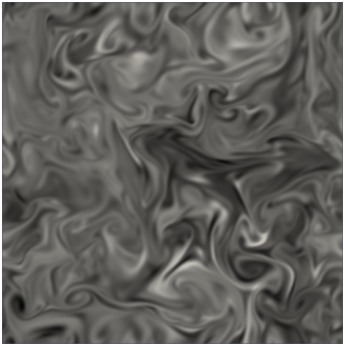
Numerical experiment #1



Numerical experiment #2



Numerical experiment #3



- Three Direct Numerical Simulations (DNS) starting from slightly different initial conditions.
- The difference in initial conditions is of the order of 0.01%
- The three solutions are shown at the same time.
- **The three fields have common features but are locally completely different.**

Turbulence is random and chaotic

Statistical description of turbulent flows

For a laminar flow:

- We can use theory (Navier-Stokes equations) to calculate the velocity u at a certain point $\boldsymbol{x}$ and time t
- We can make an experiment and measure u at $\boldsymbol{x}$ and t
- From about a century of experience, we are quite confident that the computed and measured values of u will agree and will be correct

The Navier-Stokes equations applies also to turbulent flows, however:

- Since u is random, its local and instantaneous value at $\boldsymbol{x}$ and t is inherently unpredictable
- All local and instantaneous values that we compute/measure are single **realisations** of a random phenomenon
- A theory can only aim to determine the *probability* of events
- We can determine statistical quantities (mean, variance, probability density, etc).

Definition of average

- Let consider a generic scalar function $u(\boldsymbol{x}, t)$ (e.g., one component of the velocity) which varies with the position $\boldsymbol{x}$ and the time t.
- We perform a (numerical or physical) experiment to measure u and repeat the experiment N times to measure u always at the same position and time.
- Each measured value is a **realisation**.
- A collection of realisations u_i with $i = 1, ..., N$ is an **ensemble**.
- **The average value of the realisations is the ensemble average:**

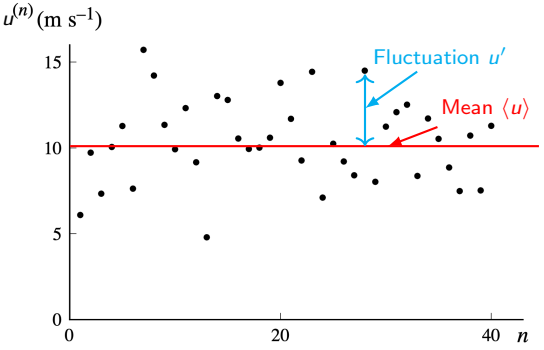
$$\langle u \rangle = \frac{1}{N} \sum_{i=1}^{N} u_i$$

- The ensemble average of an infinite number of realisations is the **expectation**:

$$E(u) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} u_i$$

- If N is sufficiently large, the ensemble average $\langle u \rangle$ is a good approximation of the expectation $E(u)$

Fluctuations and statistical moments



- We can define the fluctuation

$$u'(\mathbf{x}, t) = u(\mathbf{x}, t) - \langle u \rangle$$

- $u'(\boldsymbol{x}, t)$ and $u(\boldsymbol{x}, t)$ always varies with $\boldsymbol{x}$ and t while $\langle u \rangle$ might not.
- **The average of the fluctuation is zero**:

$$\langle u' \rangle = 0$$

- It is possible to define the **m-order statistical moments** of an ensemble:

$$\langle u^m \rangle = \frac{1}{N} \sum_{i=1}^{N} {u_i}^m \quad \leftarrow \boxed{\text{Note that: } m \text{ is an exponent, } i \text{ is the index of the sum}}$$

- The **central moment** is the moment of the fluctuations: $\langle u'^m \rangle = \frac{1}{N} \sum_{i=1}^{N} u_i'^m$

Fluctuations and high order moments

- The first central moment is the mean of the fluctuations and is zero: $\langle u' \rangle = 0$
- The second central moment is the **variance**:$\langle u'^2 \rangle$
- The third central moment is the **skewness**: $\langle u'^3 \rangle$
- The fourth central moment is the **kurtosis**: $\langle u'^4 \rangle$

- The square root of the second central moment (variance) is the **standard deviation**:

$$\sigma = \sqrt{\langle u'^2 \rangle} \leftarrow \boxed{\text{here there is the fluctuation}}$$

The square root of the second moment is the **root mean square (rms)**:

$$u_{RMS} = \sqrt{\langle u^2 \rangle} \quad \leftarrow \boxed{\text{here there is the velocity}}$$

Stationary flows

- The flow velocity u is always time dependent in a turbulent flow.
- However, the statistics of u (mean, variance, etc) might be independent of time: u is said to be **stationary** (different from steady!)
- It is possible to compute the statistics (mean, variance, etc) from time averages.
- More precisely, the time average is an estimate of the ensemble average

$$\langle u \rangle = \frac{1}{N} \sum_{i=1}^{N} u_i = \frac{1}{\Delta t} \int_0^{\Delta t} u dt$$

- **Only one long experiment is needed instead of N experiments to compute the statistics**

Homogeneous flows

- The flow velocity u is always space dependent in a turbulent flow.
- However, the statistics of u (mean, variance, etc) might be independent of space (one or more space directions): u is said to be **homogeneous** in one or more directions (different from uniform!)
- It is possible to compute the statistics (mean, variance, etc) from spatial averages, for example in a volume if the flow is homogeneous in all three directions:

$$\langle u \rangle = \frac{1}{N} \sum_{i=1}^{N} u_i = \frac{1}{V} \int_V u dV$$

- **Again, only one experiment is needed instead of N experiments to compute the statistics**

Probability Density Function - PDF

- In general, the statistics of a random variable u (the flow velocity in our case) are completely described by its **Probability Density Function (PDF)**
- PDF $f(v)$ of the velocity u: likelihood for the velocity u to have the value v in a certain point and time
- All the statistics of the velocity (mean, variance, etc) can be computed from the PDF
- For example, the mean is defined by:

$$\langle u \rangle = \int_{-\infty}^{\infty} v f(v) dv$$

- While the variance is:

$$var(u) = \langle u'^2 \rangle = \int_{-\infty}^{\infty} (v - \langle u \rangle)^2 f(v) dv$$

- And similarly for all the other statistical quantities.

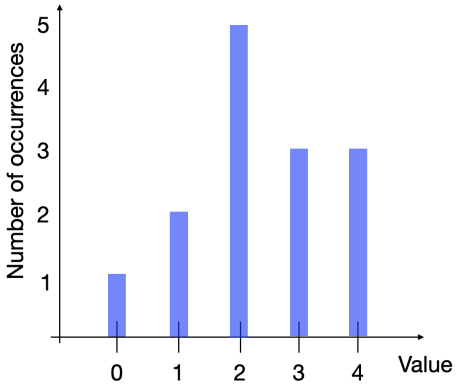
Probability Density Function - PDF

- Let's consider a list of measurements, e.g., the amount of rain (in mm) in Edinburgh in the last two weeks.
- We have 14 samples for our experiment.

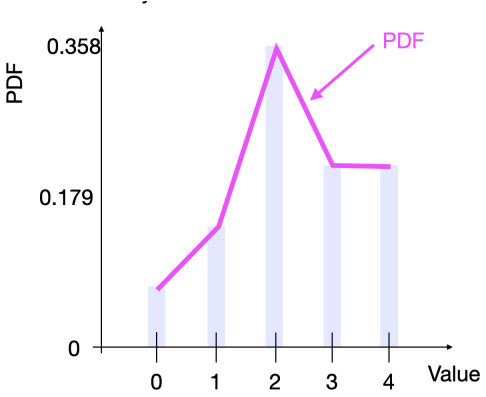
Rain data

Day	mm
1	0
2	1
3	2
4	4
5	1
6	2
7	3
8	2
9	4
10	3
11	2
12	4
13	3
14	2

Step 1: count how many time each value occurred (histogram)



Step 2: normalise so that the area under the curve is unity



Are these flows turbulent?



