Fluid Mechanics 4

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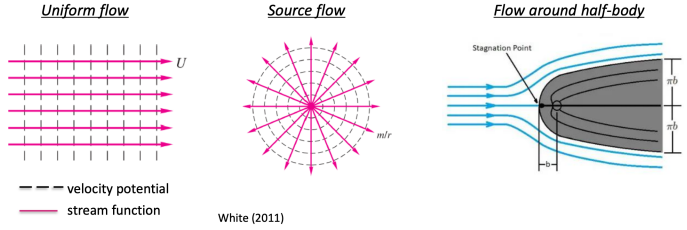
Lecture 14

Combination of elementary potential flows



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Source in Uniform Stream – Half body



White (2011)

$$\psi_{\text{half-body}} = \psi_{\text{uniform}} + \psi_{\text{source}}$$

$$\phi_{\text{half-body}} = \phi_{\text{uniform}} + \phi_{\text{source}}$$

- Cartesian:
  $\psi = Uy + \frac{q}{2\pi}\tan^{-1}\left(\frac{y}{x}\right)$
- Polar:
  $\psi = Ur\sin\theta + \frac{q}{2\pi}\theta$

- Cartesian:
  $\phi = Ux + \frac{q}{2\pi}\ln\left(\sqrt{x^2+y^2}\right)$
- Polar:
  $\phi = Ur\cos\theta + \frac{q}{2\pi}\ln(r)$

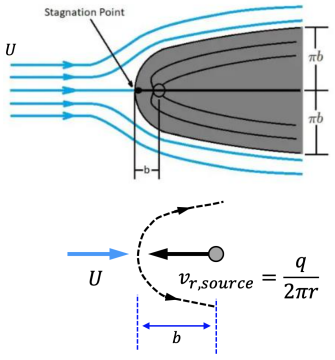
Source in Uniform Stream – Half body

At $r = b$, U and $v_{r,\text{source}}$ will oppose each other and cancel $\rightarrow$ stagnation point

- $U - v_{r,\text{source}} = 0; \quad v_{r,\text{source}} = \frac{q}{2\pi b}$
- At $r = b : v_{r,\text{source}} = \frac{q}{2\pi b} = U \rightarrow b = \frac{q}{2\pi U} \quad (*)$

Stream function at stagnation point

- $r = b;\ \theta = 180^\circ = \pi$
- $\psi_{\text{stagnation}} = Ub\sin(\pi) + \frac{q}{2\pi}\pi = \frac{q}{2}$
- From $(*) \rightarrow \frac{q}{2} = b\pi U \quad (**)$
- Combine $(**)$ and $\psi_{\text{stagnation}}$ gives equation of streamline passing through stagnation
  - $b\pi U = Ur\sin\theta + \frac{q}{2\pi}\theta$
  - $r = b(\pi - \theta)/\sin\theta$



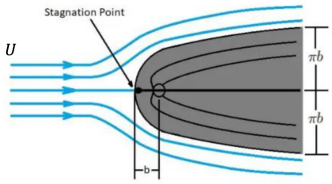
Source in Uniform Stream – Half body

Along stagnation streamline

$$r = \frac{b(\pi - \theta)}{\sin\theta}$$

Half-width: $\theta \to 0$ or $\theta \to 2\pi$

$$r = \frac{y}{\sin\theta} = \frac{b(\pi-\theta)}{\sin\theta} \to y = \pm b\pi$$

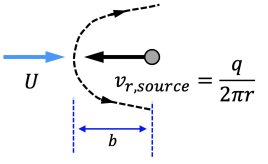


Velocity components of half body

- $v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = U\cos\theta + \frac{q}{2\pi r}$
- $v_\theta = -\frac{\partial\psi}{\partial r} = -U\sin\theta$

Velocity magnitude: $V^2 = v_r^2 + v_\theta^2$

- At any point: $V^2 = \left(U\cos\theta + \frac{q}{2\pi r}\right)^2 + (-U\sin\theta)^2 = U^2 + \frac{Uq\cos\theta}{\pi r} + \left(\frac{q}{2\pi r}\right)^2$
- With $b = \frac{q}{2\pi U} \to V^2 = U^2\left(1 + 2\frac{b}{r}\cos\theta + \frac{b^2}{r^2}\right)$



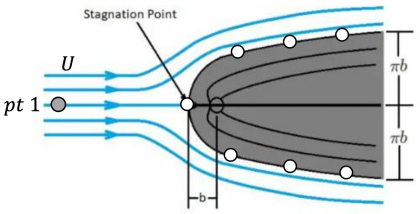
Source in Uniform Stream – Half body

With velocity V^2 known at any point, we can use Bernoulli to compute the pressure

$$p_o + \frac{1}{2}\rho U^2 + \rho g z_1 = p + \frac{1}{2}\rho V^2 + \rho g z_2$$

Pressure distribution along half-body

- $p = p_o + \frac{1}{2}\rho(U^2 - V^2) + \rho g(\Delta z)$
- $V^2 = U^2\left(1 + 2\frac{b}{r}\cos\theta + \frac{b^2}{r^2}\right)$
- $r = \frac{b(\pi-\theta)}{\sin\theta}$
- Can find press. anywhere along surface
  - May need to consider relationship between r, x, and y.



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Uniform stream and doublet – cylinder

$$\psi = \psi_{\text{uniform}} + \psi_{\text{doublet}} = Ur\sin\theta - \frac{K\sin\theta}{r} = \left(U - \frac{K}{r^2}\right) r\sin\theta = Uy - \frac{Ky}{x^2+y^2}$$

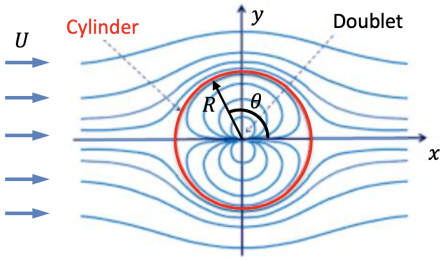
$$\phi = \phi_{\text{uniform}} + \phi_{\text{doublet}} = Ur\cos\theta + \frac{K\cos\theta}{r} = \left(U + \frac{K}{r^2}\right) r\cos\theta = Ux + \frac{Kx}{x^2+y^2}$$

Expression for K: $\psi =$ constant for $r = R$
Arbitrarily set $\psi = 0$ for $r = R$

- $\psi = \left(U - \frac{K}{r^2}\right) r\sin\theta = 0 \rightarrow K = UR^2$
- $\psi = Ur\left(1 - \frac{K}{r^2}\right)\sin\theta = Ur\left(1 - \frac{R^2}{r^2}\right)\sin\theta$
- $\phi = Ur\left(1 + \frac{K}{r^2}\right)\cos\theta = Ur\left(1 + \frac{R^2}{r^2}\right)\cos\theta$

Velocity components for stream + doublet

- $v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{\partial\phi}{\partial r} = U\left(1 - \frac{R^2}{r^2}\right)\cos\theta$
- $v_\theta = -\frac{\partial\psi}{\partial r} = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U\left(1 + \frac{R^2}{r^2}\right)\sin\theta$

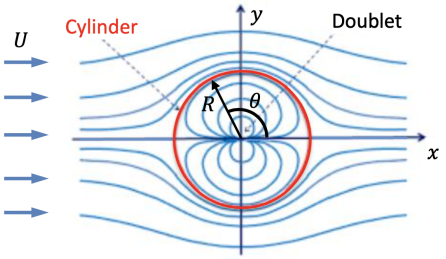


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Uniform stream and doublet – cylinder

Velocity along cylinder (i.e. $r = R$)

- $v_r = U\left(1 - \frac{R^2}{r^2}\right)\cos\theta = 0$; no radial velocity
- $v_\theta = -U\left(1 + \frac{R^2}{r^2}\right)\sin\theta = -2U\sin\theta$
- Maximum velocity around cylinder occurs at $\theta = \pm\frac{\pi}{2}$ and has magnitude of twice the uniform stream velocity



Pressure distribution along surface via Bernoulli

- $p_o + \frac{1}{2}\rho U^2 + \rho g z_1 = p_s + \frac{1}{2}\rho v_\theta^2 + \rho g z_s$
- $v_\theta = -2U\sin\theta$
- $p_s = p_o + \frac{1}{2}\rho U^2\left(1 - 4\sin^2\theta\right) + \rho g \Delta z$

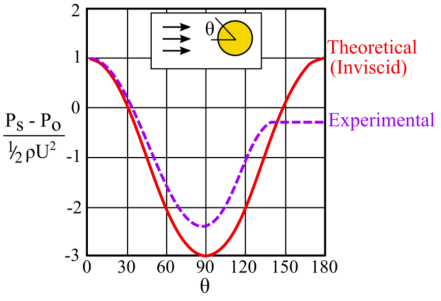
Uniform stream and doublet – cylinder

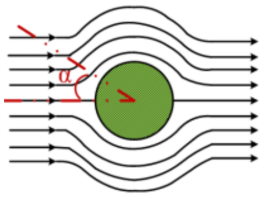
Compute drag and lift forces

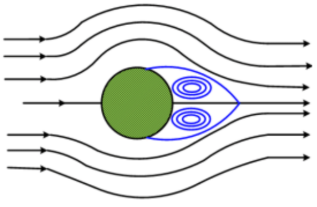
- $F_x = -\int_0^{2\pi} p_s \cos\theta \, R b d\theta$
- $F_y = -\int_0^{2\pi} p_s \sin\theta \, R b d\theta$
- Symmetric flow around cylinder: $F_x = F_y = 0$

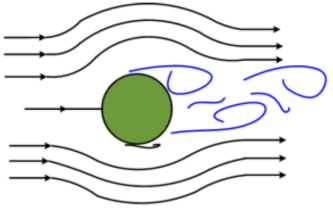
D'Alembert Paradox

- Rond d'Alembert (1717-1783). French mathematician who demonstrated that drag on bodies immersed in inviscid flow is zero.
- Contradictory to real flows: boundary layer separation and wake region









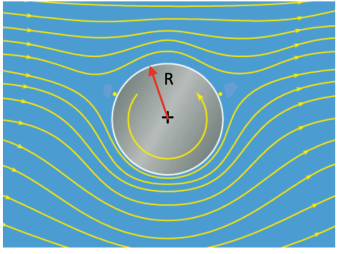
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Uniform stream, doublet, and vortex – rotating cylinder

- $\psi = Ur\left(1 - \frac{R^2}{r^2}\right)\sin\theta - \frac{\Gamma}{2\pi}\ln(r)$
- $\phi = Ur\left(1 + \frac{R^2}{r^2}\right)\cos\theta + \frac{\Gamma}{2\pi}\theta$

Velocity along the surface

- $v_{\theta S} = -\left.\frac{\partial\psi}{\partial r}\right|_{r=R} = -2U\sin\theta + \frac{\Gamma}{2\pi R}; \quad v_{rS} = \left.\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right|_{r=R} = 0$
- Stagnation location $v_{\theta S} = 0$ gives
  - $\sin\theta_{\text{stag}} = \frac{\Gamma}{4\pi UR}$



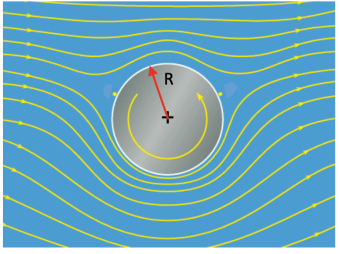
Uniform stream, doublet, and vortex – rotating cylinder

Pressure distribution via Bernoulli

- $p_o + \frac{1}{2}\rho U^2 + \rho g z_1 = p_s + \frac{1}{2}\rho v_\theta^2 + \rho g z_s$
- $p_o + \frac{1}{2}\rho U^2 + \rho g z_1 = p_s + \frac{1}{2}\rho \left(-2U\sin\theta + \frac{\Gamma}{2\pi R}\right)^2 + \rho g z_s$
- $p_s = p_o + \frac{1}{2}\rho U^2 \left(1 - 4\sin^2\theta + \frac{2\Gamma\sin\theta}{\pi U R} - \frac{\Gamma^2}{4\pi^2 R^2 U^2}\right) + \rho g \Delta z$

Forces around cylinder (b is length into page)

- $F_x = -\int_0^{2\pi} p_s \cos\theta \, R b d\theta = 0$
- $F_y = -\int_0^{2\pi} p_s \sin\theta \, R b d\theta = -\rho U b \Gamma$
- Clockwise = upwards lift;
  Counter clockwise = downward lift



Forces in potential flows

- Drag $= 0$
  This is known as as the D'Alembert's paradox, which, in a more general form, states:
  The drag force on a body moving with constant velocity relative to an incompressible and inviscid potential flow is zero.

- Lift is due to pressure and linked to the circulation Γ.
  This result is known as the **Kutta-Joukowski's theorem:**

$$\text{Lift} = -\rho U \Gamma.$$