Fluid Mechanics 4

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Lecture 13

Elementary solutions of potential flows



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Recap of definitions

- Stream function $\psi(x, y)$:

$$u = \frac{\partial \psi}{\partial y}, \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}.$$

- Velocity potential $\phi(x, y)$:

$$u = \frac{\partial \phi}{\partial x} \quad \text{and} \quad v = \frac{\partial \phi}{\partial y}.$$

- In planar polar coordinates:

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \quad \text{and} \quad u_\theta = -\frac{\partial \psi}{\partial r},$$

$$u_r = \frac{\partial \phi}{\partial r} \quad \text{and} \quad u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

- Bernoulli equation in a potential flow:

$$p + \frac{1}{2}\rho|\mathbf{u}|^2 = p + \frac{1}{2}\rho(u^2 + v^2) = p + \frac{1}{2}\rho|\nabla\phi|^2 = p + \frac{1}{2}\rho|\nabla\psi|^2 = \text{const.}$$

- The stream function and potential satisfy the equations:

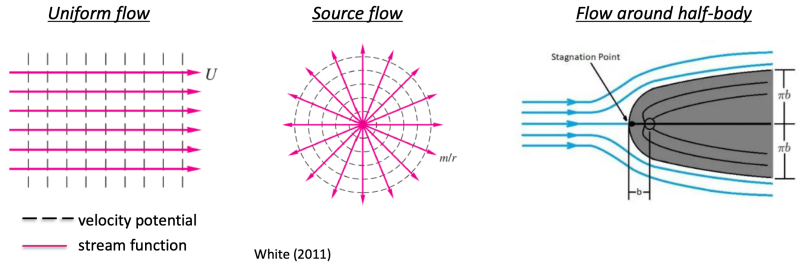
$$\nabla^2\psi = 0, \quad \text{and} \quad \nabla^2\psi = -\Gamma\delta(x - x')\delta(y - y'),$$

where δ is the Dirac delta-function, and (x', y') is the location of an ideal irrotational vortex of strength Γ.

$$\nabla^2\phi = 0, \quad \text{and} \quad \nabla^2\phi = q_s\delta(x - x')\delta(y - y'),$$

where q_s is a constant (with units of length^2/time) that sets the strength of the singularity of the source.

- **Since equations of potential and tream function are linear, any superposition of known solutions provides another solution**
- Thus, **through collection and combination, a rich variety of interesting ideal-flow solutions has emerged**.



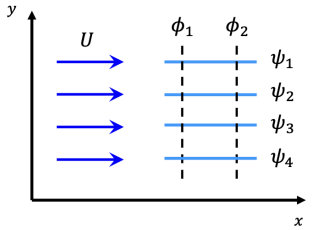
White (2011)

Uniform flow

- Uniform fluid velocity with horizontal component U and vertical component V

$$\psi = -Vx + Uy$$

$$\phi = Ux + Vy$$



Source and sink

A source or sink creates an outward or inward flow in the radial direction.
Let q be the volume flow rate emanating from/to the source/sink. Conservation of mass gives:

$$(2\pi r)v_r = q \quad \text{or:} \quad v_r = \frac{q}{2\pi r}; \quad v_\theta = 0$$

From the definition of the potential function in polar coordinates:

$$v_r = \frac{\partial \phi}{\partial r}; \quad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

$$\frac{\partial \phi}{\partial r} = \frac{q}{2\pi r}; \quad \frac{1}{r}\frac{\partial \phi}{\partial \theta} = 0$$

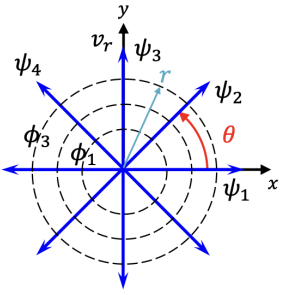
$$\phi = \frac{q}{2\pi}\ln(r)$$

Stream function gives:

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}; \quad v_\theta = -\frac{\partial \psi}{\partial r}$$

$$\frac{\partial \psi}{\partial \theta} = \frac{q}{2\pi}; \quad \frac{\partial \psi}{\partial r} = 0$$

$$\psi = \frac{q}{2\pi}\theta$$



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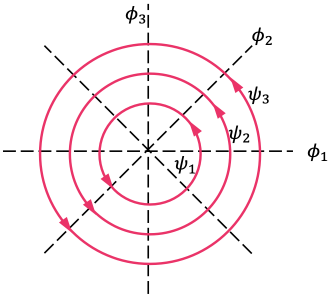
Free vortex

Streamlines are concentric circles with $v_r = 0$ and v_θ varies with r. The potential and stream function in polar coordinates for the free vortex are:

$$v_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = 0; \quad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r} = \frac{\Gamma}{2\pi r}$$

Where Γ is referred to as the circulation. Thus,

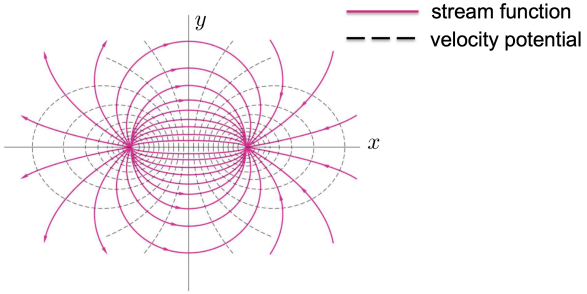
$$\phi = \frac{\Gamma}{2\pi}\theta \quad \text{and} \quad \psi = -\frac{\Gamma}{2\pi}\ln(r)$$



Source and sink at distance

Consider a source of strength Q located at $(x, y) = (-\Delta x, 0)$ and a sink with a strength of equal magnitude and opposite sign placed at $(x, y) = (\Delta x, 0)$. By superimposition, we find

$$\phi = \frac{Q}{4\pi} \ln\left(\frac{(x+\Delta x)^2 + y^2}{(x-\Delta x)^2 + y^2}\right), \quad \psi = -\frac{Q}{2\pi} \tan^{-1}\left(\frac{2\Delta x y}{x^2 + y^2 - \Delta x^2}\right)$$



Doublet

If the distance between the sink and the source approaches zero, we obtain a so-called *doublet*. The solution, after superposition and limit to zero, under the condition that $\lambda \equiv \frac{Q\Delta x}{\pi}$ remains constant, is

$$\phi = \frac{\lambda x}{x^2 + y^2}, \quad \psi = -\frac{\lambda y}{x^2 + y^2},$$

where λ is the doublet strength.

The derivation of the doublet can be found, for instance, in Kundu *et al.*.
It is important to consider the definition of the *doublet strength*. The doublet is the result of the limit for $\Delta x \to 0$ of a source and a sink with equal and opposite strength Q, at a distance $2\Delta x$. Therefore, to achieve a finite value of $\lambda \equiv \frac{Q\Delta x}{\pi}$, which is a finite, positive, scalar defining the doublet strength, Q must tend to infinity with the same speed as Δx tends to 0.
In other words,

$$\Delta x \to 0, \quad Q \to \infty,$$

and

$$Q\Delta x = \text{constant}.$$

