Fluid Mechanics 4

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Lecture 12

Introduction to ideal flows. Streamfunctions and velocity potential.

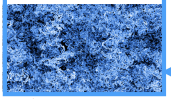


Roadmap

First principles:
$F = ma$
Mass conservation
Constitutive relations

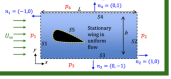
NAVIER STOKES EQUATIONS (+CONTINUITY)
⇒impossible to solve analytically

TURBULENCE
Turbulence theory
Law of the Wall



If $Re \uparrow$ the solution becomes unstable (turbulent)

Global quantities (lift, drag)



Integral equations

Hypotheses and observation to simplify the equations

Bernoulli relation between two points in the field

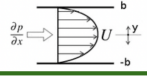
$$p_1 + \frac{1}{2}\rho V_1^2 + \rho g h_1 = p_2 + \frac{1}{2}\rho V_2^2 + \rho g h_2$$

Steady, inviscid, incompressible…

Simple geometries

$\mu = 0$
…

Analytical solutions of NS ⇒ solution of the velocity field

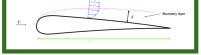


Ideal flows
Potential flows
⇒ solution of the velocity field



Close to a wall or object $\mu \neq 0$

Boundary layer theory close to the object + Potential flow far from the object



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Relevance of irrotational constant-density flows

- Constant-density fluid flowing without rotation (and no body forces):

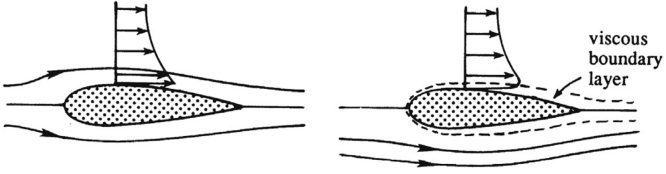
$$\nabla \cdot \mathbf{u} = 0 \quad \text{and} \quad \rho \left(\frac{D\mathbf{u}}{Dt} \right) = -\nabla p$$

  even though the fluid's viscosity μ may be non-zero.
- These are the equations of *ideal* flow.
- Useful for developing a first-cut understanding of nearly any macroscopic fluid flow
- Applications in the exterior aero- and hydrodynamics of moderate- to large-scale objects at non-trivial (large enough) sub-sonic speeds.
- Moderate size ($\mathcal{L}$) and non-trivial speed ($\mathcal{U}$) are determined jointly by the requirement that the Reynolds number, $Re = \mathcal{U}\mathcal{L}/\mu$, be large enough

Relevance of irrotational constant-density flows

- Ideal flow analysis can predict fluid velocity away from solid surfaces, surface-normal pressure forces (when the boundary layer on the surface is thin and attached).
- Ideal flow theory does not predict viscous effects like skin friction or energy dissipation, so it is not directly applicable to interior flows in pipes and ducts, to boundary-layer flows, or to any rotational flow region.
- Low-Re flows and regions of turbulence are also excluded.

- Ideal flows only satisfy the no-through-flow boundary condition on solid surfaces.
- The no-slip boundary condition is not applied, so non-zero tangential velocity at a solid surface may exist

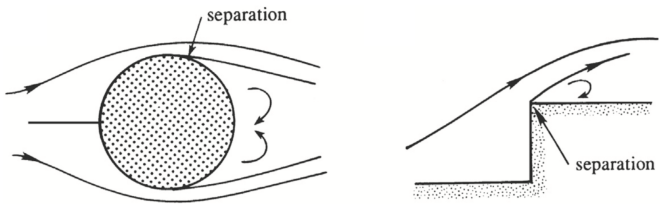


Relevance of irrotational constant-density flows

At sufficiently high Re, there are two primary differences between ideal and real flows over the same object.

1. Viscous boundary layers containing rotational fluid form on solid surfaces in the real flow, and the thickness of such boundary layers, within which viscous diffusion of vorticity is important, approaches zero as $Re \to \infty$.
2. The possible formation in the real flow of separated flow or wake regions that occur when boundary layers leave the surface on which they have developed to create a wider zone of rotational flow.

- Ideal flow theory not directly applicable to such layers or regions of rotational flow.
- However, rotational flow regions may be easy to anticipate or identify, and may represent a small fraction of a total flow field.



Stream function

- The two-dimensional incompressible continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

  is identically satisfied when u, v-velocity components are determined from a single scalar function ψ:

$$u \equiv \frac{\partial \psi}{\partial y}, \quad \text{and} \quad v \equiv -\frac{\partial \psi}{\partial x}.$$

- The function $\psi(x, y)$ is the stream function in two dimensions.
- Along a curve of $\psi = \text{constant}$, $d\psi = 0$, and this implies:

$$0 = d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = -v dx + u dy, \quad \text{or} \quad \left(\frac{dy}{dx}\right)_{\psi=\text{const}} = \frac{v}{u},$$

  which is the definition of a streamline in two dimensions.

Stream function and vorticity

- The vorticity ω_z in a flow described by ψ is:

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \omega_z = \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right) = -\nabla^2 \psi.$$

- In constant-density irrotational flow, ω_z will be zero everywhere except at the locations of ideal irrotational vortices. Thus, we are interested in solutions of:

$$\nabla^2 \psi = 0, \quad \text{and} \quad \nabla^2 \psi = -\Gamma \delta(x - x')\delta(y - y'),$$

where δ is the Dirac delta-function, and $x' = (x', y')$ is the location of an ideal irrotational vortex of strength Γ.

- In an unbounded domain, the most elementary non-trivial solutions of these equations are

1. Uniform fluid velocity with horizontal component U and vertical component V

$$\psi = -Vx + Uy$$

2. Flow induced by an irrotational vortex located at x'.

$$\psi = \frac{\Gamma}{2\pi} \ln \sqrt{(x - x')^2 + (y - y')^2},$$

Velocity potential

- Two-dimensional stream function results have been obtained by considering incompressibility first, and irrotationality second. An equivalent formulation of two-dimensional ideal flow that leads to a different scalar function is possible when incompressibility and irrotationality are considered in the other order.
- The condition of irrotationality in two dimensions is (z-component of vorticity equal to zero):

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0,$$

and it is identically satisfied when u, v-velocity components are determined from a single scalar function ϕ:

$$u \equiv \frac{\partial \phi}{\partial x} \quad \text{and} \quad v \equiv \frac{\partial \phi}{\partial y}.$$

- The function $\phi(x, y)$ is known as the velocity potential in two dimensions.
- A velocity potential exists in all irrotational flows, so such flows are frequently called potential flows.
- Curves of $\phi =$ constant are defined by:

$$0 = d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = u dx + v dy \quad \text{or} \quad \left(\frac{dy}{dx}\right)_{\phi=\text{const}} = \frac{-u}{v},$$

and are perpendicular to streamlines.

Velocity potential and incompressibility

- When using $\phi(x, y)$, the condition for incompressibility becomes:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) = \nabla^2 \phi = q(x, y).$$

- Where $q(x, y)$ is the spatial distribution of the source strength in the flow field.
- In real incompressible flows, $q(x, y) = 0$; however, ideal point sources and sinks of fluid are useful idealizations that allow the flow around objects of various shapes to be determined.
- We are interested in solutions of:

$$\nabla^2 \phi = 0, \quad \text{and} \quad \nabla^2 \phi = q_s \delta(x - x')\delta(y - y'),$$

where q_s is a constant (with units of length^2/time) that sets the strength of the singularity of the source at x'.

- In an unbounded domain, the most elementary solutions are:

1. Uniform fluid velocity with horizontal component U and vertical component V

$$\phi = Ux + Vy$$

2. Flow induced by an ideal point source of strength q_s located at x'. q_s is the source's volume flow rate per unit length perpendicular to the plane of the flow.

$$\phi = \frac{q_s}{2\pi} \ln \sqrt{(x - x')^2 + (y - y')^2},$$

Boundary Conditions in Ideal Flow

- ψ or ϕ can provide a complete description of a two-dimensional ideal flow.
- In addition, ψ is readily extended to rotational flows while ϕ is readily extended to unsteady and three-dimensional flows.
- Boundary conditions must be considered to extend the elementary ideal flow solutions to more interesting geometries.
- The boundary conditions normally encountered in irrotational flows are:
  1. **No flow through a solid surface.** As required for conservation of mass, the component of fluid velocity normal to a solid surface must equal the velocity of the boundary normal to itself. For a stationary body, this condition reduces to $\mathbf{n} \cdot \mathbf{u} = 0$, which implies:
  $$\frac{\partial \phi}{\partial n} = 0 \quad \text{or} \quad \frac{\partial \psi}{\partial s} = 0 \quad \text{on the surface,}$$
  where s is the arc-length along the surface, and n is the surface-normal coordinate.
  However, $\partial\psi/\partial s$ is also zero along a streamline. **Thus, a stationary solid boundary in an ideal flow must also be a streamline.**
  2. **Recovery of conditions at infinity.** For the typical case of a body immersed in a uniform fluid flowing in the x-direction with speed U, the condition far from the body is:
  $$\frac{\partial \phi}{\partial x} = U, \quad \text{or} \quad \frac{\partial \psi}{\partial y} = U.$$
  When $U = 0$, the fluid far from the body is said to be quiescent.

Superposition of solutions

- Historically, irrotational flow theory was developed by finding functions that satisfy the Laplace or Poisson equations

$$\nabla^2\psi = 0, \quad \text{and} \quad \nabla^2\psi = -\Gamma\delta(x-x')\delta(y-y'),$$

$$\nabla^2\phi = 0, \quad \text{and} \quad \nabla^2\phi = q_s\delta(x-x')\delta(y-y'),$$

  and then determining the boundary conditions met by those functions.
- Directly solving these equations with geometrically complicated boundary shapes using the boundary conditions typically is difficult and requires numerical techniques.
- However, **since the Laplace and Poisson equations are linear, any superposition of known solutions provides another solution**
- Note that the superposition of two or more solutions may satisfy different boundary conditions than either of the constituents of the superposition.
- Thus, **through collection and combination, a rich variety of interesting ideal-flow solutions has emerged**.

Bernoulli's Equation

- After a solution has been obtained, fluid velocity components are determined by taking derivatives of ϕ or ψ.
- The pressure in the flow is determined by conserving momentum in the form of a Bernoulli equation:

$$p + \frac{1}{2}\rho|\mathbf{u}|^2 = p + \frac{1}{2}\rho(u^2 + v^2) = p + \frac{1}{2}\rho|\nabla\phi|^2 = p + \frac{1}{2}\rho|\nabla\psi|^2 = \text{const.}$$

For quick reference, the important equations in planar polar coordinates are:

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = 0 \quad \text{(continuity)} \quad \text{and} \quad \frac{1}{r}\frac{\partial}{\partial r}(ru_\theta) - \frac{1}{r}\frac{\partial u_r}{\partial \theta} = 0 \quad \text{(irrotationality)},$$

$$u_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \quad \text{and} \quad u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r},$$

$$\nabla^2 \psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} = 0 \quad \text{and} \quad \nabla^2 \phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0.$$