Fluid Mechanics 4

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Lecture 11

Bernoulli Equation



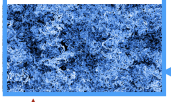
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Roadmap

First principles:
$F = ma$
Mass conservation
Constitutive relations

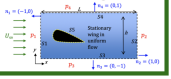
NAVIER STOKES EQUATIONS (+CONTINUITY)
⇒impossible to solve analytically

TURBULENCE
Turbulence theory
Law of the Wall



If $Re \uparrow$ the solution becomes unstable (turbulent)

Global quantities (lift, drag)



Integral equations

Hypotheses and observation to simplify the equations

Bernoulli relation between two points in the field

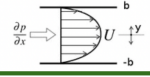
$$p_1 + \frac{1}{2}\rho V_1^2 + \rho g h_1 = p_2 + \frac{1}{2}\rho V_2^2 + \rho g h_2$$

Steady, inviscid, incompressible…

Simple geometries

$\mu = 0$
…

Analytical solutions of NS ⇒ solution of the velocity field

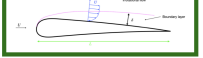


Ideal flows
Potential flows
⇒ solution of the velocity field



Close to a wall or object $\mu \neq 0$

Boundary layer theory close to the object + Potential flow far from the object



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Derivation of the Bernoulli Equation

Assumptions

- **Inviscid flow**: The fluid has no viscosity, so viscous forces are neglected.
- **Incompressible flow**: The fluid density ρ is constant.
- **Steady flow**: The velocity field does not change with time.
- **Along a streamline**: The motion is examined along s streamline

The general form of the Navier-Stokes equation for a Newtonian fluid is:

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \mu\nabla^2\mathbf{u} + \mathbf{f}$$

where:

- ρ is the density of the fluid,
- $\mathbf{u}$ is the velocity vector,
- p is the pressure,
- μ is the dynamic viscosity,
- $\mathbf{f}$ represents body forces like gravity.

Simplifications for Bernoulli's Equation

1. **Inviscid assumption**: Neglect the viscous term $\mu\nabla^2\mathbf{u}$, so the Navier-Stokes equation becomes:

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \mathbf{f}$$

2. **Steady flow**: The term $\frac{\partial \mathbf{u}}{\partial t}$ vanishes because the velocity does not change with time:

$$\rho\left(\mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \mathbf{f}$$

3. **Body forces**: Assuming the body force is due to gravity $\mathbf{f} = \rho\mathbf{g}$, we can rewrite the equation as:

$$\rho\left(\mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \rho\mathbf{g}$$

Expressing the Convective Derivative

The term $\mathbf{u} \cdot \nabla \mathbf{u}$ can be rewritten using the vector identity:

$$\mathbf{u} \cdot \nabla \mathbf{u} = \nabla \left(\frac{u^2}{2} \right) - \boldsymbol{\omega} \times \mathbf{u}$$

where:

- $u^2 = \mathbf{u} \cdot \mathbf{u}$ is the magnitude of the velocity squared,
- $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ is the **vorticity** of the flow.

Thus, the Navier-Stokes equation becomes:

$$\rho \left(\nabla \left(\frac{u^2}{2} \right) - \boldsymbol{\omega} \times \mathbf{u} \right) = -\nabla p + \rho \mathbf{g} \qquad \text{or} \qquad \rho \nabla \left(\frac{u^2}{2} \right) - \rho \boldsymbol{\omega} \times \mathbf{u} = -\nabla p + \rho \mathbf{g}$$

- The term $\omega \times \mathbf{u}$ is the cross product of the vorticity and the velocity. The vector $\omega \times \mathbf{u}$ is **perpendicular** to both vorticity ω and velocity $\mathbf{u}$.
- Along a **streamline**, the velocity vector $\mathbf{u}$ is **tangential** to the streamline by definition.
- The cross product $\omega \times \mathbf{u}$ is a vector perpendicular to the streamline. Since we are considering variations along the streamline, any component of the cross product perpendicular to the streamline does not affect the motion of the fluid along the streamline.

In other words, along a streamline:

$$\omega \times \mathbf{u} = 0$$

Simplifying the Equation

With the vorticity term vanishing along a streamline, the equation reduces to:

$$\rho\nabla\left(\frac{u^2}{2}\right) = -\nabla p + \rho\mathbf{g}$$

The gravitational body force $\rho\mathbf{g}$ can be written as the gradient of the potential energy per unit mass, $\mathbf{g} = -\nabla\Phi$, where $\Phi = gz$ (for gravitational potential in the vertical direction z):

$$\rho\nabla\left(\frac{u^2}{2}\right) = -\nabla p - \rho\nabla\Phi$$

This simplifies to:

$$\nabla\left(\frac{u^2}{2} + \frac{p}{\rho} + \Phi\right) = 0$$

Since the gradient of a scalar function being zero implies that the function is constant along a streamline, we integrate this equation to obtain the Bernoulli equation:

$$\boxed{\frac{u^2}{2} + \frac{p}{\rho} + gz = \text{constant}}$$

The Bernoulli equation is a remarkable result: It is a simple algebraic equation (it is not a differential or integral equation!!) that can provide information at different points of the flow.

Bernoulli Equation for Irrotational Flow

Assumptions

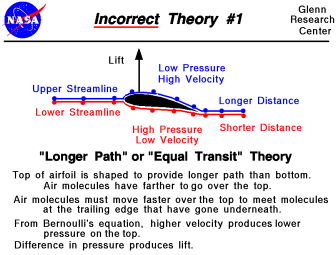
- **Inviscid flow**: The fluid has no viscosity, so viscous forces are neglected.
- **Incompressible flow**: The fluid density ρ is constant.
- **Steady flow**: The velocity field does not change with time.
- **Irrotational Flow**: Vorticity is zero. $\boldsymbol{\omega} = \nabla \times \mathbf{u} = 0$

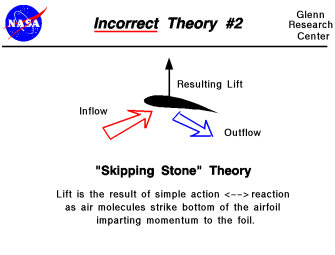
- If $\nabla \times \mathbf{u} = 0$ then $\boldsymbol{\omega} \times \mathbf{u} = 0$ everywhere (not only along a streamline) so the Bernoulli equation:

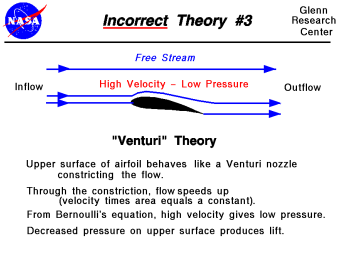
$$\boxed{\frac{u^2}{2} + \frac{p}{\rho} + gz = \text{constant}}$$

  is valid everywhere and the constant is the same everywhere.

Many "Classical" Wrong Explanations of Airfoil Lift







- Bernoulli equation is often used in combination with wrong assumptions to try to explain certain aerodynamics phenomena in a simple way.
- The application of Bernoulli equation is usually not wrong by itself, by it is applied by estimating some of the terms in the equation in a questionable (if not wrong) way.
- A famous example is shown in the left figure, using the "Longer Path" or "Equal Transit Time" theory
  - Airfoils are shaped with the upper surface longer than the bottom. The air molecules have farther to travel over the top than along the bottom in order to meet up at the trailing edge. Therefore, the molecules going over the top must travel faster than the molecules moving under the wing. Because the upper flow is faster, then, from Bernoulli's equation, the pressure is lower. The difference in pressure across the airfoil produces the lift.
  - Yes, experiments show that the flow over the top of a lifting airfoil does travel faster than the flow beneath the airfoil. But the flow is much faster than the speed required to have the molecules meet up at the trailing edge. Two molecules near each other at the leading edge will not end up next to each other at the trailing edge.