Fluid Mechanics 4

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Lecture 10

Application of Navier-Stokes Equation in Integral Form



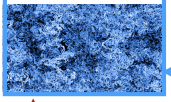
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Roadmap

First principles:
$F = ma$
Mass conservation
Constitutive relations

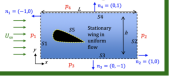
NAVIER STOKES EQUATIONS (+CONTINUITY)
⇒impossible to solve analytically

TURBULENCE
Turbulence theory
Law of the Wall



If $Re \uparrow$ the solution becomes unstable (turbulent)

Global quantities (lift, drag)



Integral equations

Hypotheses and observation to simplify the equations

Bernoulli relation between two points in the field

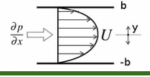
$$p_1 + \frac{1}{2}\rho V_1^2 + \rho g h_1 = p_2 + \frac{1}{2}\rho V_2^2 + \rho g h_2$$

Steady, inviscid, incompressible…

Simple geometries

$\mu = 0$
…

Analytical solutions of NS ⇒ solution of the velocity field

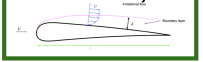


Ideal flows
Potential flows
⇒ solution of the velocity field



Close to a wall or object $\mu \neq 0$

Boundary layer theory close to the object + Potential flow far from the object



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Integral form of the Navier-Stokes equation

- Instead of focusing on a single point, the **integral form** applies conservation laws over a **control volume**.
- This is useful to compute **global flow properties**:
  - mass flux,
  - momentum balance,
  - forces (lift, drag, pressure drop).

- The mass and Navier-Stokes equations in integral form are:

$$\int_V \frac{\partial \rho}{\partial t}\, dV + \int_S \rho \boldsymbol{u} \cdot \boldsymbol{n}\, dS = 0.$$

$$\int_V \frac{\partial \rho \boldsymbol{u}}{\partial t}\, dV + \int_S \rho(\mathbf{u} \cdot \mathbf{n})\mathbf{u}\, dS = -\int_S p\mathbf{n}\, dS + \int_S \boldsymbol{\tau} \cdot \mathbf{n}\, dS + \int_V \rho \mathbf{f}\, dV$$

where:

- $\mathbf{u}$ is the velocity field,
- p is the pressure,
- $\boldsymbol{\tau}$ is the stress tensor,
- $\mathbf{f}$ represents body forces (e.g., gravity),
- V is the control volume,
- S is the control surface, and $\mathbf{n}$ is the unit normal vector to the surface.

Pressure Drop in a Channel

Task:
We aim to derive the relationship between the pressure drop in a channel and the wall shear stress using the integral form of the steady Navier-Stokes equation.

The **steady** Navier-Stokes equation in its integral form is given by:

$$\int_S \rho(\mathbf{u}\cdot\mathbf{n})\mathbf{u}\, dS = -\int_S p\mathbf{n}\, dS + \int_S \boldsymbol{\tau}\cdot\mathbf{n}\, dS + \int_V \rho\mathbf{f}\, dV$$

We assume the following:

- The flow is fully developed, meaning the velocity profile does not change along the flow direction.
- Body forces are negligible.
- The flow is incompressible with a constant density ρ, i.e., $\nabla\cdot\mathbf{u} = 0$.

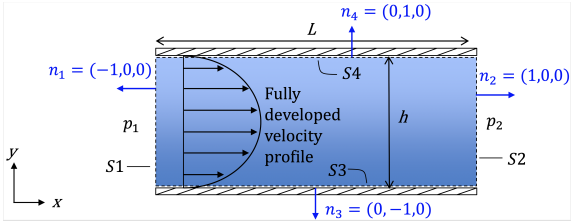
Simplifications for Fully Developed Flow

We now apply this equation to a rectangular control volume inside the channel. The control volume has:

- Length L in the flow direction (along the x-axis),
- Height h, which is the distance between the plates.

The control surfaces are:

- The inlet surface, S_1, at $x = 0$,
- The outlet surface, S_2, at $x = L$,
- The top and bottom walls at $y = \pm\frac{h}{2}$.



Since the flow is fully developed, the velocity profile does not change along the x-direction, so the momentum entering in S_1 and exiting in S_2 are the same; in addition, there is no flux across S_3 and S_4, therefore:

$$\int_S \rho(\mathbf{u}\cdot\mathbf{n})\mathbf{u}\, dS = 0$$

Thus, the integral form of the momentum equation becomes:

$$0 = -\int_S p\mathbf{n}\, dS + \int_S \boldsymbol{\tau}\cdot\mathbf{n}\, dS$$

We now consider only the x-direction and express everything per unit width of the channel ($L_z = 1$).

The pressure forces act on the inlet and outlet surfaces of the control volume. The net pressure force in the x-direction is:

$$-\int_S p\mathbf{n}\, dS = -\left(\int_{S_1}(-p_1)dS + \int_{S_2} p_2 dS\right) = p_1 \cdot h - p_2 \cdot h = -\Delta p \cdot h$$

where $\Delta p = p_2 - p_1$ is the pressure drop over the length L.
The viscous shear stress acts on the top and bottom walls of the channel. The total viscous shear force acting on the walls is:

$$\int_S \boldsymbol{\tau} \cdot \mathbf{n}\, dS = \int_{S_4} \tau_w dS + \int_{S_3} \tau_w dS = -\tau_w \cdot 2L$$

where $\tau_w = \mu \frac{\partial u}{\partial x}|_{\text{wall}}$ is magnitude of the wall shear stress, and the factor of 2 accounts for the shear forces on both the top and bottom walls.

Force Balance

Using these results in the integral equation:

$$0 = -\int_S p\mathbf{n}\, dS + \int_S \boldsymbol{\tau} \cdot \mathbf{n}\, dS$$

We find that, for steady flow, the net force due to the pressure drop must balance the viscous shear forces at the walls:

$$\Delta p \cdot h = -\tau_w \cdot 2L$$

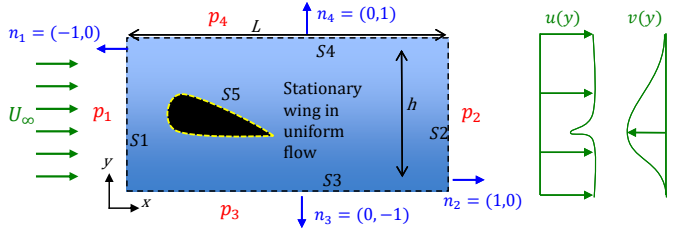
Rearranging the above equation to solve for the wall shear stress τ_w:

$$\tau_w = -\frac{\Delta p \cdot h}{2L}$$

This derivation shows that the wall shear stress τ_w is directly proportional to the pressure drop Δp over the length L of the channel. The relationship depends on the channel height h and is essential for understanding how pressure gradient drives the flow, while the shear stresses resist the flow in the channel.

Drag and Lift of Airfoil

Task:
We aim to derive an expression for the drag and lift of an airfoil in a uniform wind in open atmosphere.



- Control volume CV with **top and bottom surface along streamlines**. This avoid fluxes across these surfaces. However, in the following we will often assume that these streamlines are horizontal to simplify the problem even more. A good approximation if the size of CV in the vertical direction is large compared to the airfoil.
- Surfaces of CV are far from the airfoil; also assume that viscous forces on the 4 external surfaces are negligible.
- h is the vertical extent of the CV at the exit surface. Note that this might be different from the extent at the inlet surface.
- The drag and lift are the forces that the fluid exerts on the airfoil, in the direction of the incoming wing and along its normal, respectively.
- Therefore, the airfol exerts opposite forces on the fluid. These forces are the integral of all the stresses on the surface of the airfol (surface $S5$).

Aifoil Drag

- For the drag D, we use the integral momentum equation in the x-direction for a steady incompressible flow with constant density:

$$\int_S \rho(\mathbf{u}\cdot\mathbf{n})\mathbf{u}\,dS = -\int_S p\mathbf{n}\,dS + \int_S \boldsymbol{\tau}\cdot\mathbf{n}\,dS$$

- We write these 3 terms over the 5 surfaces $S1$, ... $S5$, assuming that the length of the airfoils in the z direction is b.

Momentum flux (LHS of the equation)

$$\int_S \rho(\mathbf{u}\cdot\mathbf{n})\mathbf{u}\,dS$$

- $S1$: $-\rho U_\infty^2 bh$
- $S2$: $\rho b\int_0^h u(y)^2 dy$
- $S3$ and $S4$: zero (no flux since along a streamline)
- $S5$: zero (no flux since it is a wall)

Pressure term

$$-\int_S p\mathbf{n}\,dS$$

- $S1$: $p_1 S1$
- $S2$: $-p_2 S2$
- $S3$ and $S4$: zero if we assume the surface are horizontal.
- $S5$: $-D_{\text{pressure}}$

If we assume that the vertical size of the domain does not change much and the pressure is atmospheric in both $S1$ and $S2$, then $p_1 S1 - p_2 S2 = 0$.

Viscous terms

$$\int_S \boldsymbol{\tau}\cdot\mathbf{n}\,dS$$

- $S1$–$S4$: zero since we assume that viscous forces can be neglected there.
- $S5$: $-D_{\text{viscous}}$

Putting all together: $D = D_{\text{pressure}} + D_{\text{viscous}} = \rho U_\infty^2 bh - \rho b\int_0^h u(y)^2 dy = \rho U_\infty^2 bh - \rho U_\infty^2 b\int_0^h (u(y)/U_\infty)^2 dy$

Aifoil Lift

- For the lift $\mathcal{L}$, we use the integral momentum equation in the y-direction for a steady incompressible flow with constant density:

$$\int_S \rho(\mathbf{u}\cdot\mathbf{n})\mathbf{u}\,dS = -\int_S p\mathbf{n}\,dS + \int_S \boldsymbol{\tau}\cdot\mathbf{n}\,dS$$

- We write these 3 terms over the 5 surfaces $S1$, ... $S5$, assuming that the length of the airfoils in the z direction is b.

Momentum flux (LHS of the equation)

$$\int_S \rho(\mathbf{u}\cdot\mathbf{n})\mathbf{u}\,dS$$

- $S1$: zero (no vertical velocity)
- $S2$: $\rho b\int_0^h v(y)u(y)dy$
- $S3$ and $S4$: zero (no flux since along a streamline)
- $S5$: zero (no flux since it is a wall)

Pressure term

$$-\int_S p\mathbf{n}\,dS$$

- $S1$, $S2$: zero (normal is in x direction)
- $S3$: $b\int_0^L p_3 dx$
- $S4$: $-b\int_0^L p_4 dx$
- $S5$: $-\mathcal{L}_{\text{pressure}}$

Viscous terms

$$\int_S \boldsymbol{\tau}\cdot\mathbf{n}\,dS$$

- $S1$–$S4$: zero since we assume that viscous forces can be neglected there.
- $S5$: $-\mathcal{L}_{\text{viscous}}$

Putting all together: $\mathcal{L} = \mathcal{L}_{\text{pressure}} + \mathcal{L}_{\text{viscous}} = b\int_0^L (p_3 - p_4)dx - \rho b\int_0^h v(y)u(y)dy$