Fluid Mechanics 4 (Mechanical)
Formula Sheet

When necessary assume the following values

- Air density: $\rho = 1\,\text{kg m}^{-3}$.
- Air kinematic viscosity: $\nu = 1.5 \times 10^{-5}\,\text{m}^2\text{s}^{-1}$.
- Speed of sound in air: $c_s = 3.4 \times 10^2\,\text{m s}^{-1}$.
- Water density: $\rho = 10^3\,\text{kg m}^{-3}$.
- Water kinematic viscosity: $\nu = 1.0 \times 10^{-6}\,\text{m}^2\text{s}^{-1}$.
- Transition Reynolds number for a smooth pipe: $3 \times 10^3 - 5 \times 10^4$.
- Transition Reynolds number for a smooth sphere: 5×10^5.
- Transition Reynolds number for a smooth airfoil boundary layer: $5 \times 10^5 - 5 \times 10^6$.

Fundamental Theorem of Calculus

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Gradient Theorem

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}_2) - f(\mathbf{r}_1)$$

Green's Theorem

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

Stokes' Theorem

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

Gauss's Theorem (Divergence Theorem)

$$\oint_{\partial V} \mathbf{F} \cdot d\mathbf{A} = \iiint_V (\nabla \cdot \mathbf{F})\,dV$$

Acceleration

$$\boldsymbol{a}(\boldsymbol{x}, t) = \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = \frac{D\boldsymbol{u}}{Dt}$$

Velocity gradient decomposition

$$\nabla \boldsymbol{u} = \frac{1}{3}(\nabla \cdot \boldsymbol{u})\boldsymbol{I} + \boldsymbol{E} + \boldsymbol{\Omega}$$

$$\boldsymbol{E} = sym(\boldsymbol{A}) = \frac{1}{2}(\boldsymbol{A} + \boldsymbol{A}^T)$$

$$\boldsymbol{\Omega} = skw(\boldsymbol{A}) = \frac{1}{2}(\boldsymbol{A} - \boldsymbol{A}^T)$$

$$\boldsymbol{A} = \begin{pmatrix} \frac{\partial u_x}{\partial x} - \frac{\nabla \cdot \boldsymbol{u}}{3} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} - \frac{\nabla \cdot \boldsymbol{u}}{3} & \frac{\partial u_y}{\partial z} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z} - \frac{\nabla \cdot \boldsymbol{u}}{3} \end{pmatrix}$$

Strain rate tensor

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)$$

Reynolds Transport Theorem

$$\frac{dB(t)}{dt} = \int_{V_0} \frac{\partial(\rho b)}{\partial t} dV + \int_{S_0} (\rho b)\boldsymbol{u} \cdot \boldsymbol{n} dS$$

$$\frac{dB(t)}{dt} = \int_{V_0} \frac{\partial(\rho b)}{\partial t}\,dV + \int_{V_0} \nabla \cdot (\rho b \boldsymbol{u})\,dV$$

Continuity Equation

$$\int_{V_0} \frac{\partial \rho}{\partial t}\,dV + \int_{S_0} \rho \boldsymbol{u} \cdot \boldsymbol{n}\,dS = 0.$$

$$\frac{\partial \rho}{\partial t} + \boldsymbol{u} \cdot \nabla \rho + \rho \nabla \cdot \boldsymbol{u} = 0$$

Momentum Equation

$$\int_{V_0} \frac{\partial \rho \boldsymbol{u}}{\partial t}\, dV + \int_{S_0} \rho \boldsymbol{u}\boldsymbol{u} \cdot \boldsymbol{n}\, dS = -\int_{S_0} p\boldsymbol{I} \cdot \boldsymbol{n}\, dS + \int_{S_0} \boldsymbol{\tau} \cdot \boldsymbol{n}\, dS + \int_{V_0} \rho \boldsymbol{f}\, dV,$$

$$\frac{\partial(\rho \boldsymbol{u})}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}\boldsymbol{u}) = -\nabla p + \nabla \cdot \boldsymbol{\tau} + \rho \boldsymbol{f}$$

$$\rho \frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + \rho f_i$$

Constitutive equation

$$T_{ij} = -p\delta_{ij} + \tau_{ij} = -p\delta_{ij} + 2\mu\left(S_{ij} - \frac{1}{3}S_{mm}\delta_{ij}\right) + \mu_v S_{mm}\delta_{ij}.$$

Stokes assumption

$$\mu_v = \lambda + 2\mu/3 = 0$$

Stress Tensor

$$\mathbf{T} = \begin{pmatrix} -p + 2\mu\frac{\partial u}{\partial x} + \left(\mu_v - \frac{2}{3}\mu\right)\nabla \cdot \mathbf{u} & \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \\ \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & -p + 2\mu\frac{\partial v}{\partial y} + \left(\mu_v - \frac{2}{3}\mu\right)\nabla \cdot \mathbf{u} & \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) & -p + 2\mu\frac{\partial w}{\partial z} + \left(\mu_v - \frac{2}{3}\mu\right)\nabla \cdot \mathbf{u} \end{pmatrix}$$

Navier-Stokes Equation (with the Stokes assumption)

$$\rho\left(\frac{\partial u_j}{\partial t} + u_i\frac{\partial u_j}{\partial x_i}\right) = -\frac{\partial p}{\partial x_j} + \rho f_j + \frac{\partial}{\partial x_i}\left[\mu\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right) - \frac{2}{3}\mu\frac{\partial u_m}{\partial x_m}\delta_{ij}\right]$$

Navier-Stokes Equation for incompressible flow with constant viscosity

$$\rho\left(\frac{\partial u_j}{\partial t} + u_i\frac{\partial u_j}{\partial x_i}\right) = -\frac{\partial p}{\partial x_j} + \rho f_j + \mu\frac{\partial^2 u_j}{\partial x_i \partial x_i}$$

Poisseuille flow

$$u_x(y) = \frac{1}{2\mu}\left(-\frac{dp}{dx}\right)\left(hy - y^2\right)$$

Couette flow

$$u_x(y) = \frac{U}{h}y$$

Bernoulli equation

$$\frac{u^2}{2} + \frac{p}{\rho} + gz = \text{constant}$$

Stream function

$$u = \frac{\partial \psi}{\partial y} \quad v = -\frac{\partial \psi}{\partial x}$$

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \quad u_\theta = -\frac{\partial \psi}{\partial r}$$

Velocity Potential

$$u = \frac{\partial \phi}{\partial x} \quad v = \frac{\partial \phi}{\partial y}$$

$$u_r = \frac{\partial \phi}{\partial r} \quad u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

Laplace Equation

$$\nabla^2 \phi = 0, \quad \nabla^2 \psi = 0$$

Uniform potential flow

$$\phi = Ux + Vy \quad \text{and} \quad \psi = -Vx + Uy$$

Source/Sink

$$\phi = \frac{q}{2\pi}\ln(r) \quad \text{and} \quad \psi = \frac{q}{2\pi}\theta$$

Potential vortex

$$\phi = \frac{\Gamma}{2\pi}\theta \quad \text{and} \quad \psi = -\frac{\Gamma}{2\pi}\ln(r)$$

Potential Flow Past a Rotating Cylinder

$$\psi = Ur\left(1 - \frac{R^2}{r^2}\right)\sin\theta - \frac{\Gamma}{2\pi}\ln(r)$$

$$\phi = Ur\left(1 + \frac{R^2}{r^2}\right)\cos\theta + \frac{\Gamma}{2\pi}\theta$$

Kutta-Joukowski's Theorem

$$L = -\rho U \Gamma.$$

Ensemble average and fluctuations

$$\langle U \rangle = \frac{1}{N}\sum_{i=1}^{N} u_i \quad \text{and} \quad u'(\mathbf{x}, t) = u(\mathbf{x}, t) - \langle U \rangle$$

Variance, standard deviation, and root mean square

$$u_{\text{VAR}} = \langle u'^2 \rangle \quad \sigma = \sqrt{\langle u'^2 \rangle} \quad u_{RMS} = \sqrt{\langle u^2 \rangle}$$

Mean continuity equation

$$\nabla \cdot \langle \mathbf{u} \rangle = 0$$

Mean Navier-Stokes equation

$$\frac{\partial \langle u_j \rangle}{\partial t} + \langle u_i \rangle \frac{\partial \langle u_j \rangle}{\partial x_i} = -\frac{1}{\rho}\frac{\partial \langle p \rangle}{\partial x_j} + \nu \nabla^2 \langle u_j \rangle - \frac{\partial}{\partial x_i}\left\langle u_i' u_j' \right\rangle$$

Kinetic energy of the mean flow

$$\tilde{E} = \frac{1}{2}\langle \mathbf{u} \rangle \cdot \langle \mathbf{u} \rangle$$

Turbulent kinetic energy

$$k = \frac{1}{2}\langle \mathbf{u}' \cdot \mathbf{u}' \rangle = \frac{1}{2}\langle u_i' u_i' \rangle$$

Turbulent viscosity hypothesis

$$-\rho \left\langle u_i' u_j' \right\rangle + \frac{2}{3}\rho k \delta_{ij} = \rho \nu_T(\mathbf{x}, t)\left(\frac{\langle \partial u_i \rangle}{\partial x_j} + \frac{\langle \partial u_j \rangle}{\partial x_i}\right)$$

Prandtl's Mixing Length Turbulence Model

$$\nu_T = l_m^2 \left| \frac{\partial \langle u \rangle}{\partial x} \right|$$

Mean kinetic energy equation

$$\frac{\partial \tilde{E}}{\partial t} + \langle \mathbf{u} \rangle \cdot \nabla \tilde{E} + \nabla \cdot \langle \mathbf{T} \rangle = -\bar{\varepsilon} - \mathcal{P}$$

$$\langle T_i \rangle = \langle u_j \rangle \left\langle u_i' u_j' \right\rangle + \langle u_i \rangle \frac{\langle p \rangle}{\rho} - 2\nu \langle u_j \rangle \langle S_{ij} \rangle$$

$$\bar{\varepsilon} = 2\nu \langle S_{ij} \rangle \langle S_{ij} \rangle$$

$$\langle S_{ij} \rangle = \frac{1}{2}\left(\frac{\partial \langle u_i \rangle}{\partial x_j} + \frac{\partial \langle u_j \rangle}{\partial x_i}\right)$$

$$\mathcal{P} = -\left\langle u_i' u_j' \right\rangle \frac{\partial \langle u_i \rangle}{\partial x_j}$$

Turbulent kinetic energy

$$\frac{\partial k}{\partial t} + \langle \mathbf{u} \rangle \cdot \nabla k + \nabla \cdot \mathbf{T}' = \mathcal{P} - \varepsilon$$

$$T_i' = \frac{1}{2}\left\langle u_i' u_j' u_j' \right\rangle + \frac{\langle u_i' p' \rangle}{\rho} - 2\nu \langle u_j s_{ij} \rangle$$

$$\varepsilon = 2\nu \langle s_{ij} s_{ij} \rangle$$

$$s_{ij} = \frac{1}{2}\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)$$

$$\mathcal{P} = -\left\langle u_i' u_j' \right\rangle \frac{\partial \langle u_i \rangle}{\partial x_j}$$

Two point, one time covariance

$$R_{ij}(\mathbf{r}, \mathbf{x}, t) = \langle u_i(\mathbf{x}, t) u_j(\mathbf{x} + \mathbf{r}, t) \rangle$$

Spectra

$$\Phi_{ij}(\mathbf{k}, t) = \frac{1}{(2\pi)^3}\iiint_{-\infty}^{\infty} e^{-i\mathbf{k}\cdot\mathbf{r}} R_{ij}(\mathbf{r}, t) d\mathbf{r}$$

Energy spectrum in isotropic flow

$$E(k, t) = \iiint_{-\infty}^{\infty} \frac{1}{2}\Phi_{ij}(\mathbf{k}, t)\delta(|\mathbf{k}| - k) d\mathbf{k}$$

Turbulent kinetic energy in wavenumber space

$$\mathcal{E} = \int_0^\infty E(k, t) dk = \frac{1}{2}R_{ii}(0, t) = \frac{1}{2}\langle u_i' u_i' \rangle$$

Kolmogorov variables

$$\eta = (\nu^3/\varepsilon)^{1/4}$$

$$u_\eta = (\varepsilon\nu)^{1/4}$$

$$\tau_\eta = (\nu/\varepsilon)^{1/2}$$

Ratio between the largest and the smallest scales of turbulence

$$\eta / l_0 \sim Re^{-3/4}$$

Kolmogorov spectrum in the inertial range

$$E(k) = C\varepsilon^{2/3}k^{-5/3} \quad \text{with} \quad C = 1.5$$

Wall units

$$u_\tau = \sqrt{\frac{\tau_w}{\rho}}$$

$$\delta_\nu = \frac{\nu}{u_\tau}$$

$$u^+ = \frac{\langle u \rangle}{u_\tau} \quad \text{and} \quad y^+ = \frac{u_\tau}{\nu} y = \frac{y}{\delta_\nu}$$

$$Re_\tau = \frac{u_\tau \delta}{\nu} = \frac{\delta}{\delta_\nu}$$

Linear and log laws in the turbulent boundary layer

$$u^+ = y^+$$

$$u^+ = \frac{1}{k}\ln\left(y^+\right) + B$$

Boundary layer thickness

$$\frac{\delta(x)}{x} \approx \frac{1}{\sqrt{Re_x}} \quad \text{or} \quad \delta(x) \approx \sqrt{\frac{\nu x}{U}} \quad \text{with} \quad Re_x = Ux/\nu$$

Wall normal velocity in boundary layer

$$v \approx \frac{U}{\sqrt{Re_x}}$$

Wall shear stress in boundary layer

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} \approx \mu \frac{U}{\delta} = \mu \frac{U}{x}\sqrt{Re_x}$$

Local skin friction coefficient in boundary layer

$$c_f(x) = \frac{\tau_w}{\frac{1}{2}\rho U^2} \approx \frac{2}{\sqrt{Re_x}}$$

Drag coefficient in boundary layer

$$C_D = \frac{1}{L}\int_0^L c_f(x)dx = 2c_f(L) \approx \frac{4\nu}{UL}$$

Displacement thickness

$$\delta_1 = \int_{y=0}^{\infty} \left(1 - \frac{u}{U}\right) dy$$

Momentum thickness

$$\delta_2 = \int_{y=0}^{\infty} \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$$

$$D = \rho U^2 \delta_2 \quad \text{so} \quad \delta_2 = \frac{D}{\rho U^2} \quad \text{or} \quad \frac{\delta_2}{L} = \frac{1}{2}C_D$$

$$\tau_w = \frac{d}{dx}\left(\rho U^2 \delta_2\right) = \rho U^2 \frac{d\delta_2}{dx} \quad \text{and} \quad c_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = 2\frac{d\delta_2}{dx}$$

Turbulent boundary layer equation

$$\langle u \rangle \frac{\partial \langle u \rangle}{\partial x} + \langle v \rangle \frac{\partial \langle u \rangle}{\partial y} = U_e \frac{\partial U_e}{\partial x} + \nu \frac{\partial^2 \langle u \rangle}{\partial y^2} - \frac{\partial}{\partial y}\langle u'v' \rangle$$

Blasius solution for the laminar boundary layer

$$\frac{\delta}{x} = \frac{5.0}{Re_x^{1/2}} \qquad \frac{\delta_1}{x} = \frac{1.721}{Re_x^{1/2}} \qquad \frac{\delta_2}{x} = \frac{0.665}{Re_x^{1/2}}$$

$$H_{12} = \frac{\delta_1}{\delta_2} = 2.59$$

$$c_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{0.664}{Re_x^{1/2}}$$

$$C_D = \frac{1}{L}\int_0^L c_f dx = 2c_f(L) = \frac{1.328}{Re_L^{1/2}}$$

1/7 power Law

$$\frac{u}{U} = \left(\frac{y}{\delta}\right)^{1/7}$$

$$\frac{\delta}{x} = \frac{0.16}{Re_x^{1/7}} \qquad \delta_1 = \frac{1}{8}\delta \qquad \delta_2 = \frac{7}{72}\delta \qquad H_{12} = 1.3$$

$$c_f = \frac{0.027}{Re_x^{1/7}}$$

$$C_D = \frac{1}{L}\int_0^L c_f dx = \frac{7}{6}c_f(L) = \frac{0.031}{Re_L^{1/7}}$$

Boundary layer growth (laminar and turbulent)

$$\frac{\delta}{x} = \frac{5.0}{Re_x^{1/2}} \qquad \frac{\delta}{x} = \frac{0.16}{Re_x^{1/7}}$$