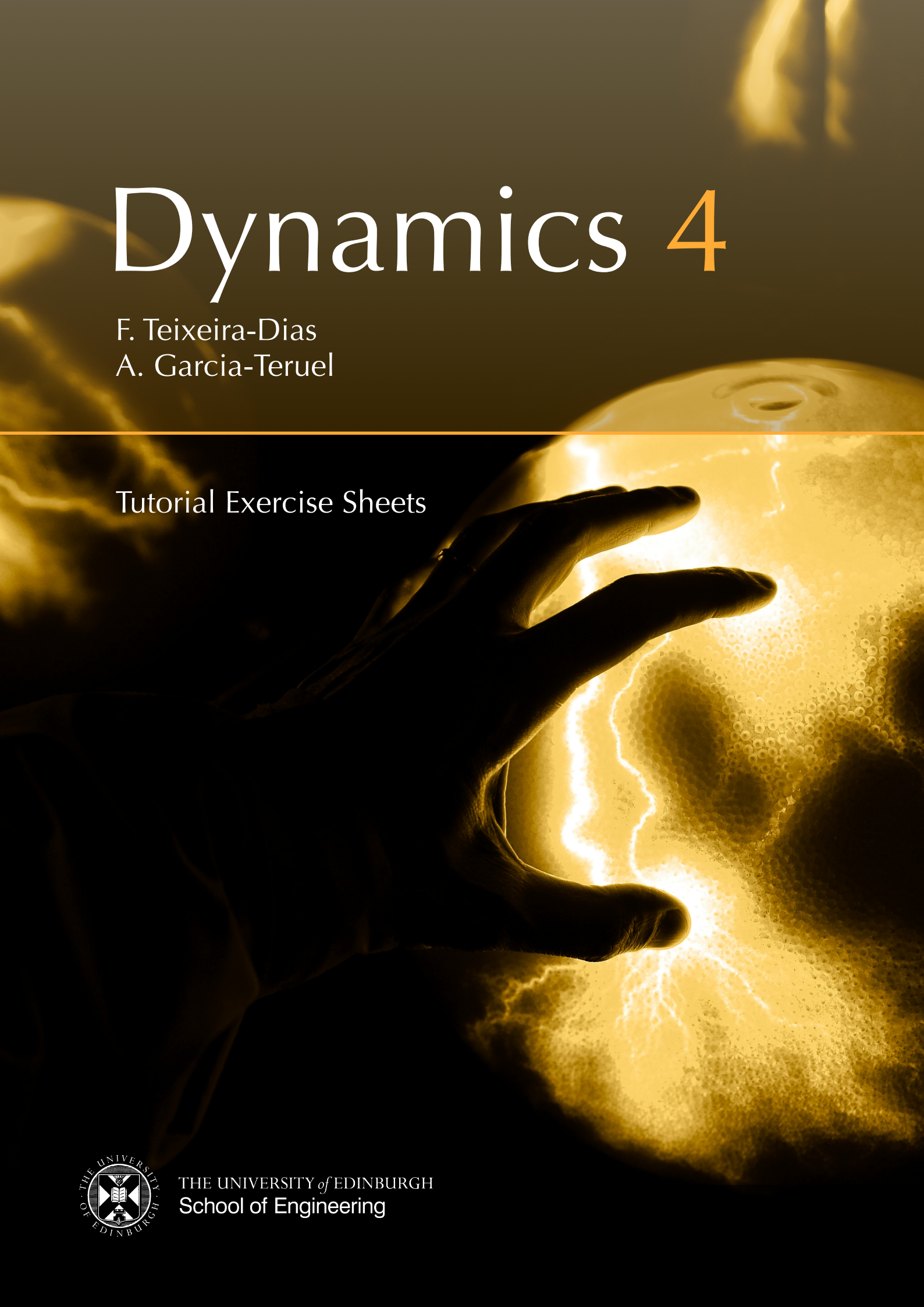
Dynamics 4

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Tutorial Exercise Sheets



THE UNIVERSITY of EDINBURGH
School of Engineering

Dynamics 4

Exercise Sheet 1

Notes for this Exercise Sheet The topics covered in this exercise sheet are: (i) deriving and understanding equations of motion of a dynamical system; (ii) Using Lagrange's equations as a straightforward approach to obtain the equations of motion; (iii) determine the kinetic and potential energy expressions for different types of dynamical systems; (iv) analyse dynamic systems with different characteristics (e.g. linear and rotation movements, extension and torsion elastic potential energy. etc.) All these topics were discussed in the lectures and are covered in the lecture notes. If you want to read more about this or go through more examples you can, for example, see Sections 6.6 and 6.7 (pages 619-624) on the reference suggested for further reading[a]. Additionally, you can solve problems 6.1 to 6.6 on pages 675 and 676 on this same reference.

[a] S.S. Rao, Mechanical Vibrations (6^{th} edition, SI), Pearson, 2018 (ISBN 978-1-292-17860-8).

Dynamics 4
MECE10002
Semester 1, 2026-27

Classroom
Classroom 12, Alrick

Learning module
LM1

Course notes
Section 2.1
Section 2.2

Slides
LM1.3 to LM1.20

Discussion thread
analytical_dynamics

Question 1.1

The system in Figure 1 is a pendulum with an imposed vertical motion of the pivot. This oscillatory motion is $\bar{z}(t) = A\cos(\omega t)$. Although strictly this is not a conservative system (the potential energy is time-dependent) it is a special case and the conservative form of Lagrange's equations can in fact be applied. Determine the system's differential equations of motion using Lagrange's method.

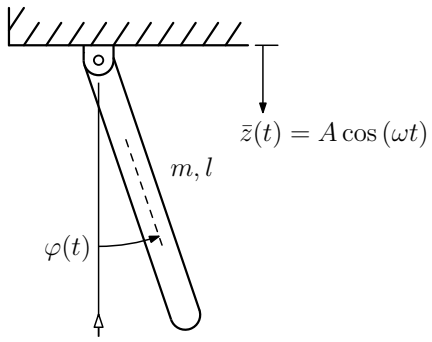


Figure 1: Dynamic system for Question 1.1

Solution

The equation of motion for this system is:

$$\frac{ml^2}{3}\ddot{\varphi} - \frac{ml}{2}\ddot{\bar{z}}\sin(\varphi) + \frac{mgl}{2}\sin(\varphi) = 0$$



Question 1.2

Consider the dynamic system shown in Figure 2. AB is a rigid uniform bar of mass M, pivoted at point A, with an added attached mass m as shown. Assume the angular deflection of AB is small and neglect the gravitational potential energy.

a) Determine the system's kinetic and potential energy expressions.
b) Determine the corresponding equations of free vibration (equations of motion).
c) Explain in what circumstances it is a reasonable assumption to neglect gravitational potential energy in this case.

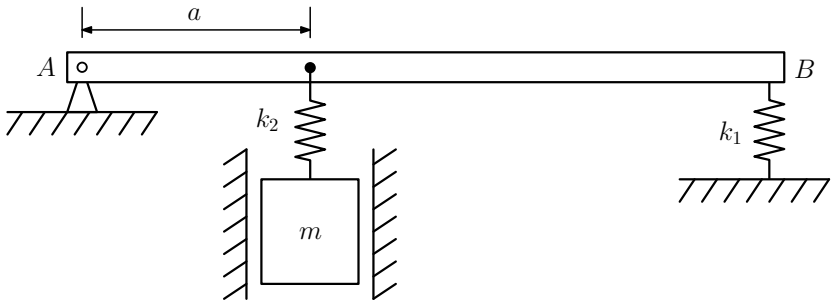


Figure 2: Dynamic system for Question 1.2

Solution

Neglecting gravitational potential energy and considering that $\theta(t)$ is the rotation of the bar and $x(t)$ is the vertical displacement o the mass, the equations of motion for this system are:

$$m\ddot{x} - k_2 a\theta + k_2 x = 0$$
$$I_A\ddot{\theta} + \left(k_1 L^2 + k_2 a^2\right)\theta - k_2 a x = 0$$

Question 1.3

The mass M in the system shown in Figure 3 is restrained by springs and has a pendulum bar hanging from it. Neglect friction and determine the energy expressions and the system's differential equations of motion. Show that the motion of the mass and the angular motion of the pendulum are coupled (through non-linear terms). Identify these non-linear terms and obtain the linearised equations of motion of the system.

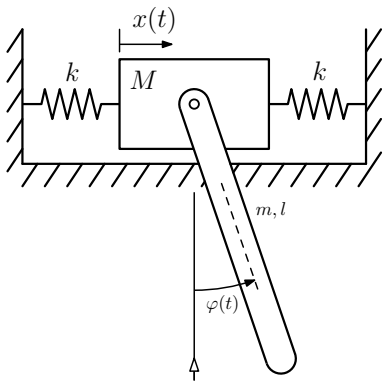


Figure 3: Dynamic system for Question 1.3

Solution

The linearised equations of motion for this system are then

$$(M+m)\ddot{x} + \frac{ml}{2}\ddot{\varphi} + 2kx = 0$$

$$\frac{ml^2}{3}\ddot{\varphi} + \frac{ml}{2}\ddot{x} + \frac{mgl}{2}\varphi = 0$$

Question 1.4

Figure 4 shows a torsional vibration system. A motor drives a centrifugal pump impeller through a reduction gear. $I_1 = 25$ kgm^2, $I_2 = 15$ kgm^2 and $I_3 = 35$ kgm^2 are the inertias of the motor armature, the gear wheel and the combined gear wheel and pump rotor, respectively. The gear ratio is $3.5:1$ ($\beta = 3.5$). The power transmission shaft has a torsional stiffness $K_\mathrm{s} = 20\times10^6$ Nm (Nm/rad). The pump has 8 blades and runs at 5250 rpm.

a) Use Lagrange's method to get the system's differential equations for torsional vibration, i.e. its equations of motion.

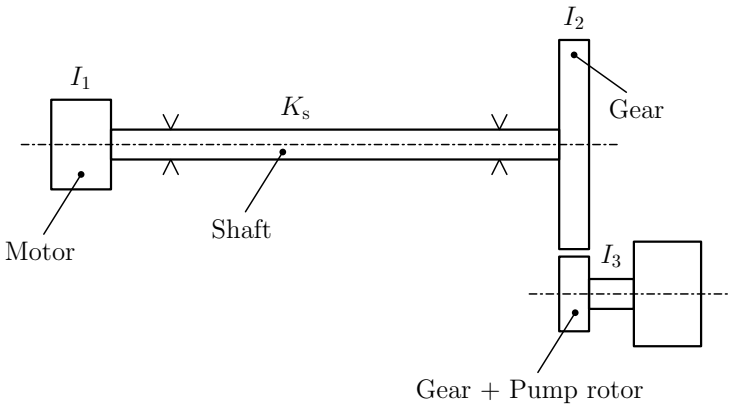


Figure 4: Dynamic system for Question 1.4

Solution

The equations of motion of the system are

$$\begin{cases} I_1\ddot{\theta}_1 + K_\mathrm{s}\theta_1 - K_\mathrm{s}\theta_2 = 0 \\ \left(I_2 + \beta^2 I_3\right)\ddot{\theta}_2 - K_\mathrm{s}\theta_1 + K_\mathrm{s}\theta_2 = 0 \end{cases}$$

and the resonance motor speed is 313 rev/min.

Dynamics 4

Exercise Sheet 2

Notes for this Exercise Sheet The topics covered in this exercise sheet are: (i) derive and understand equations of motion of dynamical systems; (ii) use Lagrange's equations to obtain the equations of motion for non-conservative dynamic systems; (iii) determine generalised forces for non-conservative systems. All these topics were discussed in the lectures and are covered in the lecture notes. If you want to read more about this or go through more examples you can, for example, see Sections 6.6 and 6.7 (pages 619-624) on the reference suggested for further reading[a]. Additionally, you can solve example 6.18 (page 656) and problems 6.1 to 6.6 on pages 675 and 676 of this same reference.

[a] S.S. Rao, Mechanical Vibrations (6^{th} edition, SI), Pearson, 2018 (ISBN 978-1-292-17860-8).

Dynamics 4
MECE10002
Semester 1, 2026-27

Classroom
Classroom 12, Alrick

Learning module
LM2

Course notes
Section 2.3

Slides
LM2.3 to LM2.10

Discussion thread
analytical_dynamics

Question 2.1

The system in Figure 1 is very similar to the one on Question 1.4. However, there is a damper and a constant force P applied to the end of the pendulum so that it is always acting along the centre line of the pendulum[1].

a) Find any generalised forces that need to be considered.
b) Determine the equations of motion of the system (e.g. by comparison with the solution from Question 1.4).

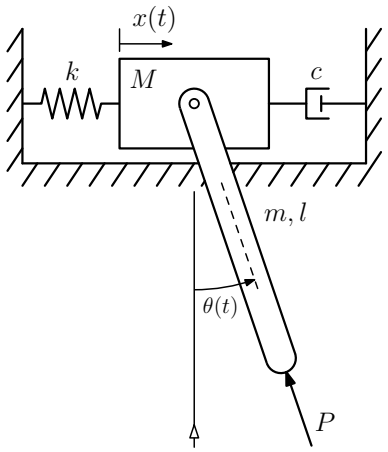


Figure 1: Dynamic system for Question 2.1.

Solution

The equations of motion for this system are:

$$(M+m)\ddot{x} + \frac{1}{2}ml\ddot{\theta} + kx = -c\dot{x} - P\theta$$

$$\frac{1}{3}ml^2\ddot{\theta} + \frac{1}{2}ml\ddot{x} + \frac{1}{2}mgl\theta = 0$$

[1] This is termed a *follower force* and could, for example, represent the thrust force due to a jet of fluid flowing out of a nozzle on the pendulum.

Question 2.2

Consider the dynamic system shown in Figure 2, similar to the one in Question 1.1. AB is a rigid uniform bar, pivoted at point A, with an added attached mass m as shown. Now there is a dynamic disturbing force $P(t) = P_0 \cos(\omega t)$ acting on the bar and a damper attached at the end (point B). Once again assume the angular deflection of AB is small and can thus be neglected. Ignore the gravitational potential energy and complete the equations of motion from Question 1.1 with the generalised forces.

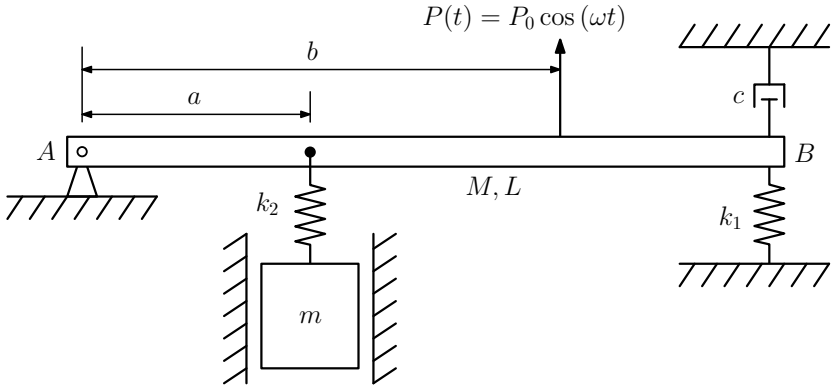


Figure 2: Dynamic system for Question 2.2.

Solution

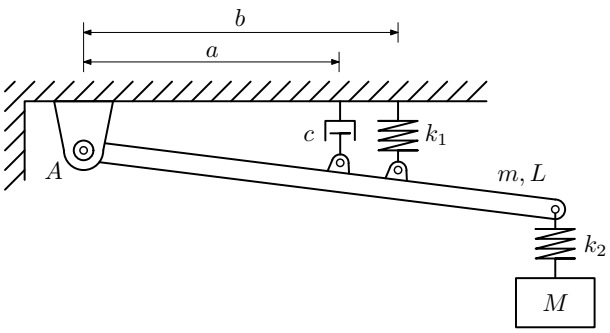
The equations of motion for this system are

$$m\ddot{x} - k_2 a\theta + k_2 x = 0$$
$$I_A\ddot{\theta} + \left(k_1 L^2 + k_2 a^2\right)\theta - k_2 a x = -cL^2\dot{\theta} + bP_0 \cos(\omega t)$$

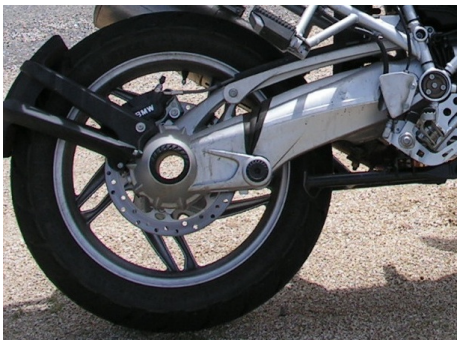
Question 2.3

The dynamic system shown in Figure 3a is a conceptual model of the motorbike suspension in Figure 3b[2]. The suspension arm has length L and mass m, and is pivoted at point A. The damping constant and the stiffness of the suspension are c and k_1, respectively. The mass of the wheel is M and the stiffness of the tyre is k_2. Neglect friction at the pivot point.

a) Identify the most appropriate generalised coordinates of the system and derive the corresponding kinetic and potential energy functions.
b) Find the generalised forces acting on the system.
c) Obtain the equations of motion of the system and linearise them if appropriate.



(a)



(b)

Figure 3: Motorbike suspension for Question 2.3.

Solution

The linearised equations of motion are

$$M\ddot{y} + k_2 y - k_2 L\theta + Mg = 0$$

and

$$\frac{1}{3}mL^2\ddot{\theta} + \frac{1}{2}mgL + k_1 b^2\theta - k_2 Ly + k_2 L^2\theta = -ca^2\dot{\theta}$$

[2]Photo licensed under the Creative Commons Attribution 2.0 Generic License (CC BY 2.0).

Question 2.4

You can use Lagrange's approach to get the equations of motion of a mass particle orbiting the Earth (e.g. a satellite). Start with the representation in figure 4, a moving mass m in free space with its position given by polar coordinates r and θ, both time functions.

a) Assume generic radial and tangential forces F_r and F_t, respectively, and use Lagrange's method to determine the equations of motion in r and θ.

b) Adapt these equations for a satellite orbiting outside the Earth's atmosphere. Assume $F_t = 0$ (no drag) and $F_r = -GMm/r^2$, that is, a radial inward force obeying Newton's inverse square law of gravitation. $M = 5.97 \times 10^{24}$ kg is the mass of the Earth and $G = 6.674 \times 10^{-11}$ N/(m/kg)^2 is the Universal Gravitational constant.

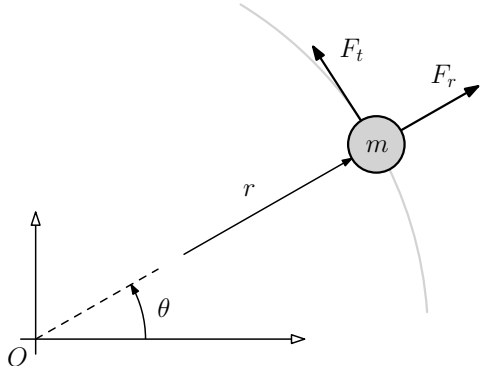


Figure 4: Orbiting satellite representation (Question 2.4).

Solution

The equations of motion for the generic system are:

$$m\ddot{r} - mr\dot{\theta}^2 = F_r$$
$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = rF_t$$

The adapted equations of motion for the orbiting satellite system are:

$$\ddot{r} - r\dot{\theta}^2 = -\frac{GM}{r^2}$$
$$r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} = 0$$

Question 2.5

A bungee jumper wishes to try something new (!). He wants to start with the bungee cord strung out horizontally from an attachment and swing down in an arc. Obviously the cord is elastic and will extend during the swing. Consider the dynamic model for this system shown in figure 5. AB is a uniform rigid bar pivoted at A and with negligible mass. Mass m represents the jumper who slides on the bar (no friction) supported from A by the bungee cord (spring stiffness K and initial length a). Obtain the equations of motion of the system considering it is released from the horizontal position at time $t = 0$.

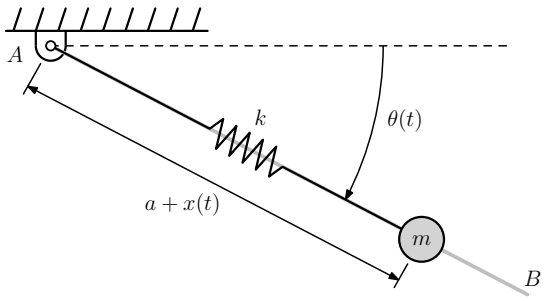


Figure 5: Dynamic model of a bungee jumper, for Question 2.5.

Solution

The equations of motion for this system are:

$$m(a+x)^2\ddot{\theta} + 2m(a+x)\dot{x}\dot{\theta} - mg(a+x)\cos\theta = 0$$
$$m\ddot{x} - m(a+x)\dot{\theta}^2 + kx - mg\sin\theta = 0$$

Dynamics 4

Exercise Sheet 3

Notes for this Exercise Sheet The topics covered in this exercise sheet are the following: (i) analysis of continuous systems; dynamic stability (and instability). All these topics were discussed in the lectures and are covered in the lecture notes. If you want to read more about this or go through more examples you can, for example, see Sections 8.1 to 8.5 (pages 745 to 788) on the reference suggested for further reading[a]. Additionally, you can solve example 6.18 (page 656) and problems 8.22 to 8.47 on pages 809 to 811 of this same reference.

[a] S.S. Rao, Mechanical Vibrations (6^{th} edition, SI), Pearson, 2018 (ISBN 978-1-292-17860-8).

Dynamics 4
MECE10002
Semester 1, 2026-27

Classroom
Classroom 12, Alrick

Learning module
LM3

Course notes
Section 2.4

Slides
LM3.3 to LM3.12

Discussion thread
dynamic_stability

Question 3.1

A disc of negligible mass is spinning horizontally at ω rad/s. A mass M can slide frictionless along a slot that runs along the diameter of the disc as shown in Figure 1. The mass is held by two equal length and stiffness springs.

a) Derive the kinetic and potential energy functions for the system.
b) Use Lagrange's method to obtain an equation of motion for the radial motion of the mass.
c) Find a steady-state solution for this equation and determine its stability.
d) How does the mass behaviour change as the rotation speed is slowly increased?

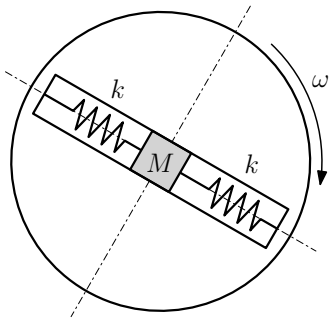


Figure 1: Rotating disc for Question 3.1.

Solution

The equation of motion of the system becomes

$$M\ddot{r} + \left(2k - M\omega^2\right) r = 0$$

For $\omega^2 < \frac{2k}{M}$, the root is imaginary and the solution is stable.

Question 3.2

The upright pendulum system shown in figure 2a can be used to model the dynamics of the vertical orientation of the behaviour of a spinning plate on a pole, such as the one shown in figure 2b. The pole has length l and mass m, and the plate is a lumped mass M fixed at the top of the pole.

a) Write the equation of motion of the upright pendulum system in figure 2a neglecting the boundary motion $\bar{z}(t)$ and all dissipative terms, such as friction.
b) Determine if the pendulum is stable or unstable in the upright position shown in figure 2.
c) Discuss the effect that increasing the length of the pole l or increasing the mass of the plate M will have on the dynamic stability of the system.
d) Write the equation of motion of the system considering the ground motion $\bar{z}(t)$ and discuss what effect this has on the stability of the pendulum in the upright position.

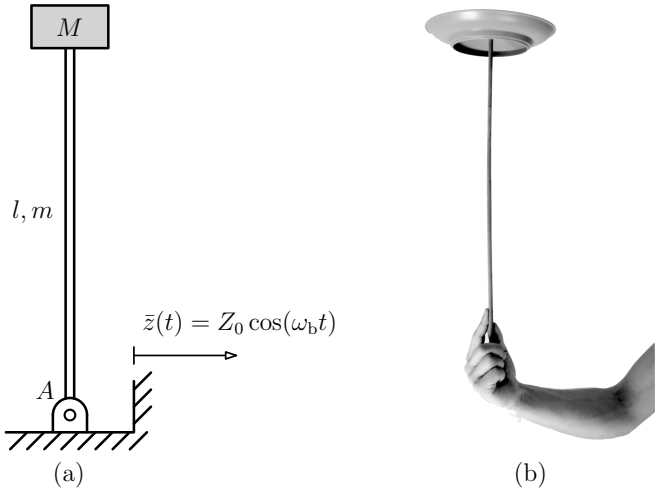


Figure 2: Spinning a plate on a pole for Question 3.2: (a) dynamic model and (b) spinning plate [Photo licence CC BY-SA 3.0].

Solution

a) The equation of motion of the upright pendulum neglecting the boundary motion $\bar{z}(t)$ and all dissipative terms is:

$$\left(M + \frac{m}{3}\right) l^2 \ddot{\theta} - \left(M + \frac{m}{2}\right) gl \sin\theta = 0$$

b) The pendulum is unstable in the upright position.
d) The equation of motion of the system considering the ground motion $\bar{z}(t)$ is:

$$\left(M + \frac{m}{3}\right) l^2 \ddot{\theta} + \left(M + \frac{m}{2}\right) l \left(\ddot{\bar{z}} \cos\theta - g \sin\theta\right) = 0$$

Question 3.3

A rigid bar AB is pivoted at point A and is mounted at the centre of a horizontal rotating disc as shown in figure 3.

a) Obtain the equation of motion for this system.
b) Find the corresponding steady-state solutions and analyse the stability of the system for different rotation velocities of the disc.

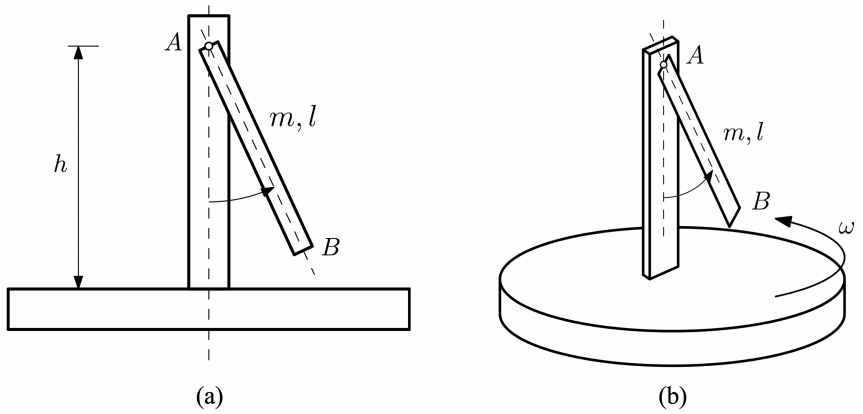


Figure 3: Rotating disc and oscillating bar system: (a) side and (b) 3D view.

Solution

a) The equation of motion of the system is

$$\frac{2}{3}ml^2\ddot{\theta} - \frac{1}{2}ml^2\omega^2\cos\theta\sin\theta + mgl\sin\theta = 0$$

b) The steady-state solution $\theta^s = \pi$ is always unstable. The steady-state solution $\theta^s = 0$ may be stable or unstable.

Question 3.4

The homogeneous L-bracket AOB shown in figure 4 has mass $2m$, is pivoted at point O and is connected to a fixed wall through a spring of stiffness k and a damper of damping constant c.

a) Obtain the equation of motion for this system.
b) Analyse the stability of the system if the spring is removed.

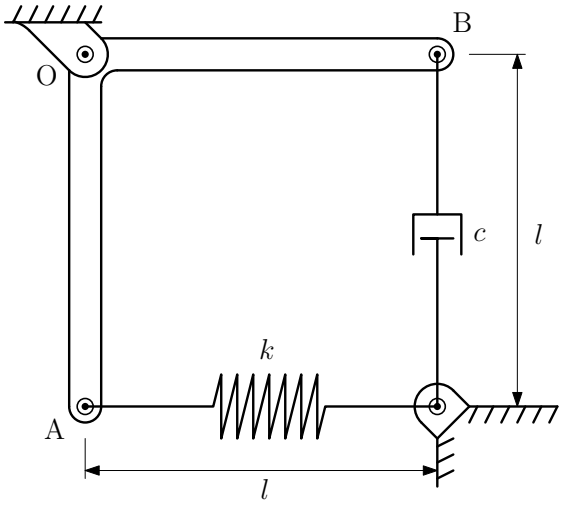


Figure 4: Oscillating bracket.

Solution

a) The equation of motion of the system is

$$\frac{2}{3}ml^2\ddot{\theta} + \frac{1}{2}mgl\left(\cos\theta + \sin\theta\right) + kl^2\sin\theta\cos\theta = -cl^2\dot{\theta}\cos\theta$$

b) When the spring is removed, regardless of the mass m, size l or damping constant c, the system will always be stable, albeit converging to the stationary position differently, which should be expected given the configuration of the L-bracket system.



Question 3.5

Consider the dynamic system in Figure 5. The rigid distributed mass pendulum AB has length L and mass m, and is pinned at point A. Mass M is connected to fixed boundaries through an elastic spring with stiffness k and a viscous damper with damping constant c. Force $P(t) = P_0 \cos(\omega t)$ is always horizontal and acts at point B, as shown in Figure 5.

a) Derive the equations of motion of the system and linearise them, assuming small vibrations. Do not neglect the effects of gravity.
b) Assume mass M is now fixed, and show that the pendulum in the hanging down position is borderline stable.

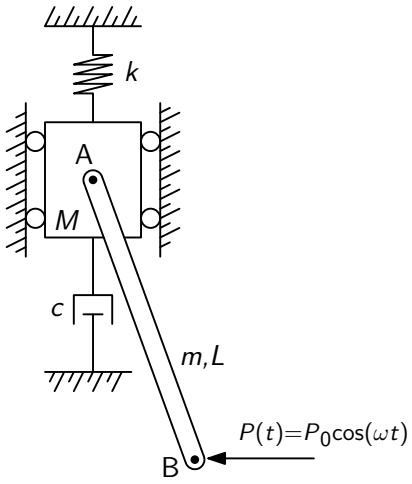


Figure 5: Dynamic system with oscillating mass and distributed mass pendulum.

Solution

The equations of motion of the system are:

$$(M+m)\ddot{y} + \frac{1}{2}mL\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right) + ky + (M+m)g = -c\dot{y}$$

$$\frac{1}{3}mL^2\ddot{\theta} + \frac{1}{2}mL\ddot{y}\sin\theta + \frac{1}{2}mgL\sin\theta = -LP_0\cos(\omega t)\cos\theta$$

Dynamics 4

Exercise Sheet 4

Notes for this Exercise Sheet This exercise sheet covers: (i) analysis of continuous systems; (ii) shape functions and energy functions for continuous systems; (iii) Lagrange's method to obtain the equations of motion of continuous systems; (iv) generalised forces for continuous systems. All these topics were discussed in the lectures and are covered in the lecture notes. If you want to read more about this or go through more examples you can, for example, see Sections 8.1 to 8.5 (pages 745 to 788) on the reference suggested for further reading[a].

[a] S.S. Rao, Mechanical Vibrations (6th edition, SI), Pearson, 2018 (ISBN 978-1-292-17860-8).

Dynamics 4
MECE10002
Semester 1, 2026-27

Classroom
Classroom 12, Alrick

Learning module
LM4

Course notes
Section 2.5

Slides
LM4.3 to LM4.28

Discussion thread
continuous_systems

Question 4.1

Figure 1 shows an uniform cantilever beam. Obtain the approximate expression for the fundamental frequency of the beam using the static deflection polynomial

$$\varphi_{\mathrm{s}}(x) = \frac{2Lx^2 - x^3}{2L^3}$$

as the assumed mode shape function. Check if this complies with all relevant boundary conditions and compare with the result obtained for a quarter-cosine function

$$\varphi_{\mathrm{d}}(x) = 1 - \cos\left(\frac{\pi x}{2L}\right)$$

Which one is the best mode shape function? Why?

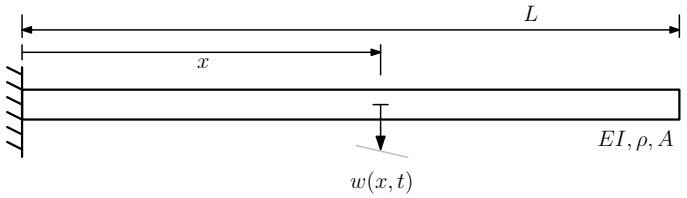


Figure 1: Cantilever beam for Question 4.1.

Solution

The natural frequency obtained with the polynomial mode shape function is

$$\omega_0^2 = 14.5\frac{EI}{mL^3}$$

The natural frequency obtained with the trigonometric mode shape function is

$$\omega_0^2 = 13.4\frac{EI}{mL^3}$$

Question 4.2

Figures 2(a) and (b) show two cases of distributed pressure load that is suddenly applied to the top surface of a uniform cantilever beam. Using the same mode shape functions as in the previous question, estimate the ratio of the maximum tip deflections produced by the two cases. Think about the generalised forces acting in both cases and try to keep calculations to a minimum.

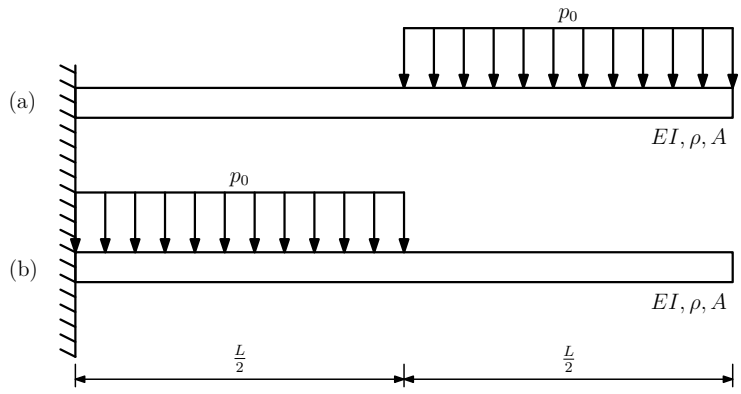


Figure 2: Cantilever beam with applied distributed pressure loads.

Solution

The ratio of the tip deflections is equal to the ratio of the generalised forces, which is

$$\frac{w_a}{w_b} = \frac{Q_a}{Q_b} = \frac{67}{13} = 5.15$$

using the polynomial function from Question 3.1 or

$$\frac{w_a}{w_b} = \frac{Q_a}{Q_b} = 6.29$$

using the cosine function from Question 3.1.



Question 4.3

A uniform beam is supported by end springs and has a damper attached at its midpoint as shown in Figure 3. Obtain the generalised force terms that must be added to the beam equations using the following mode shape functions:

$$\varphi_1(x) = 1$$

$$\varphi_2(x) = \cos\left(\frac{\pi x}{L}\right)$$

Write down the equations of motion of the system.

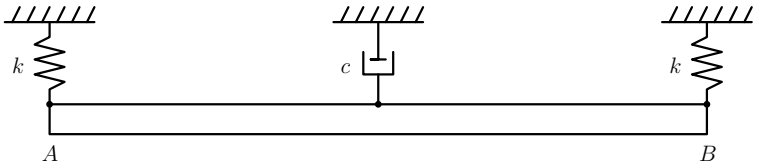


Figure 3: Beam supported on springs for Question 4.3.

Solution

The generalised force on the first and second coordinates are

$$Q_1 = Q_2 = -c(\dot{q}_1 + \dot{q}_2)$$



Question 4.4

The diving board shown in Figure 4 is modelled as a beam with a clamped end at $x = 0$ and a free end at $x = l$, where a diver with mass M stands. The mass of the board is $m = 30$ kg, the length is $l = 2$ m and the bending stiffness is $EI = 2 \times 10^4$ Nm^2.

a) Specify a shape function $\varphi(x)$ to describe the vibration behaviour of the lowest frequency mode of the system. Give reasons for your choice.
b) Using this shape function evaluate the corresponding kinetic and potential energy functions.
c) Use Lagrange's equations to obtain the equation of motion (for the lowest frequency mode).
d) Determine the mode frequency of the board alone and considering a diver standing on its free end ($M = 80$ kg).

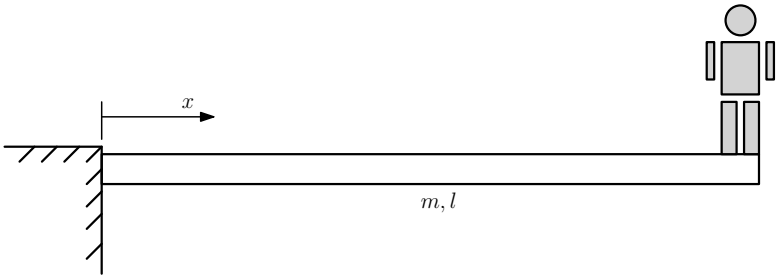


Figure 4: Diving board model for Question 4.4.

Solution

The equation of motion is

$$\left(\frac{m}{5} + M\right)\ddot{q} + \frac{4EI}{l^3}q + \left(\frac{m}{3} + M\right)g = 0$$

The mode frequencies of the board are 10.8 rad/s and 40.8 rad/s with and without the mass M, respectively.

Dynamics 4

Exercise Sheet 5

Notes for this Exercise Sheet The topics covered in this exercise sheet are: (i) dynamics of continuous systems; (ii) wave equations; (iii) longitudinal waves on a uniform bar; (iv) travelling wave solutions; (v) standing-wave solutions. All these topics were discussed in the lectures and are covered in the lecture notes. If you want to read more about this or go through more examples you can, for example, see Sections 8.1 to 8.5 (pages 745 to 788) on the reference suggested for further reading[a]. Additionally, you can solve example 6.18 (page 656) and problems 8.22 to 8.47 on pages 809 to 811 of this same reference.

[a] S.S. Rao, Mechanical Vibrations (6^{th} edition, SI), Pearson, 2018 (ISBN 978-1-292-17860-8).

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longitudinal_waves

Question 5.1

Follow the method of the bar example done in class to find expressions for the set of natural frequencies and wave lenghts of a wire of length L tightly stretched between two fixings. Wire has tension T and mass per unit length μ.

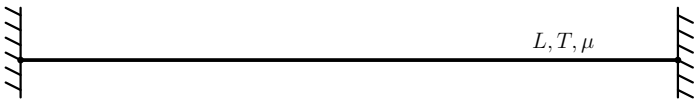


Figure 1: Stretched wire for Question 5.1.

Solution

The natural frequencies of the wire are

$$\omega_n = n\frac{\pi}{L}\sqrt{\frac{T}{\mu}} \quad \text{with} \quad n = 1, \dots$$

and the mode shapes (wave lengths)

$$\lambda_n = 2\pi\frac{c}{\omega_n} = \frac{2L}{n} \quad \text{with} \quad n = 1, \dots$$



Question 5.2

A steel bar has length $L = 3.5$ m and diameter $d = 45$ mm. This bar is suspended horizontally from two long wires fixed to the ceiling so that both ends of the bar are free to move. A small accelerometer (with negligible mass) is fixed at one end and the opposite end of the bar is tapped with a hammer. What frequencies would you expect to be registered by the accelerometer. List the first four in Hz and consider $E = 207$ GPa and $\rho = 7830$ kg/m^3.

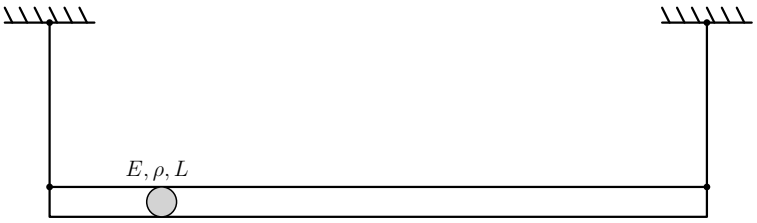


Figure 2: Suspended steel bar for Question 5.2.

Solution

The first four frequencies and wavelengths are

n	ω_n [rad/s]	ω_n [Hz]	λ_n [m]
1	4615	734.5	7.00
2	9230	1469.0	3.50
3	13845	2203.5	2.33
4	18461	2938.2	1.75

Question 5.3

A uniform bar of cross-section area A, length l and Young's modulus E is connected at both ends by springs, viscous dampers and masses as shown in Figure 3. State the appropriate boundary conditions for this system.

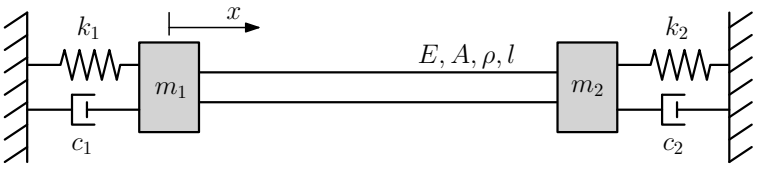


Figure 3: Boundary conditions for Question 5.3.

Solution

The boundary conditions for the bar are

$$AE\frac{\partial u}{\partial x}(0,t) = k_1 u(0,t) + c_1\frac{\partial u}{\partial t}(0,t) + m_1\frac{\partial^2 u}{\partial t^2}(0,t).$$

and

$$-AE\frac{\partial u}{\partial x}(L,t) = k_2 u(L,t) + c_2\frac{\partial u}{\partial t}(L,t) + m_2\frac{\partial^2 u}{\partial t^2}(L,t).$$



Question 5.4

Using a standing-wave approach, find the free-vibration solution and natural frequencies of a bar of length l fixed at one end and free at the other (see Figure 4).

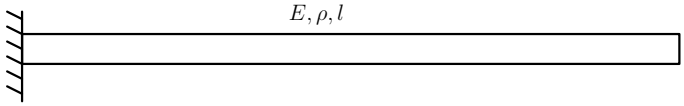


Figure 4: Fixed bar for Question 5.4.

Solution

The natural frequencies of the bar are

$$\omega_n = \frac{(1+2n)}{2}\frac{\pi c}{l} \quad \text{with} \quad n = 0, \dots$$



Question 5.5

Consider a bar of length l with one end fixed and a mass M attached at the other end, as in Figure 5. Find the first and second natural frequencies of the bar as a function of the ratio β between the mass of the bar (m) and the mass M (that is, $\beta = m/M$).

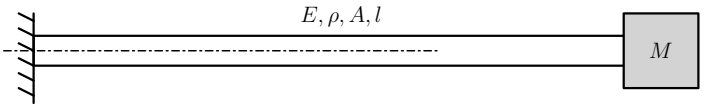


Figure 5: Fixed bar with attached mass for Question 5.5.

Solution

Mass ratio β	$\alpha_1 = \omega_1 l/c$	$\alpha_2 = \omega_2 l/c$
0.01	0.1	3.1448
0.1	0.3113	3.1736
1	0.8602	3.4267
10	1.4291	4.3063
100	1.5549	4.6658



Question 5.6

A bar of uniform cross-sectional area A, density ρ, Young's modulus E and length l is fixed at one end and free at the opposite end. This bar is subjected to an axial force F_0 at its free end, as shown in Figure 6. Study the vibrations of the bar if the force F_0 is suddenly removed.

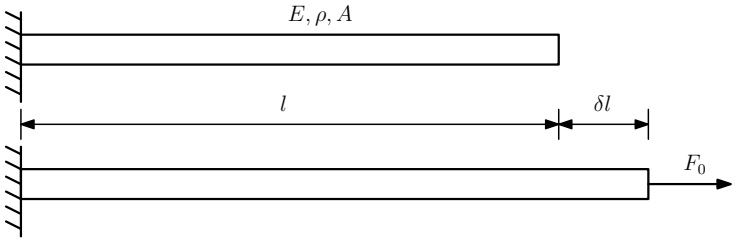


Figure 6: Fixed bar with axial force for Question 5.6.

Solution

The natural frequencies can be obtained from the cosine function

$$\omega_n = \frac{(2n+1)\pi c}{2l} \quad \text{with} \quad n = 0, \dots$$



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Dynamics 4

Exercise Sheet 6

Notes for this Exercise Sheet The topics covered in this exercise sheet are: (i) dynamics of continuous systems; (ii) wave equations; (iii) transverse (flexural) waves on a uniform beam; (iv) travelling and standing-wave solutions. All these topics were discussed in the lectures and are covered in the lecture and class notes. If you want to read more about this or go through more examples you can, for example, see Section 8.5 (pages 767 to 788) on the reference suggested for further reading[a]. Additionally, you can go through examples 8.7 to 8.9 (pages 774 and 780) and solve problems 8.30 to 8.55 on pages 809 to 813 of this same reference.

[a] S.S. Rao, Mechanical Vibrations (6^{th} edition, SI), Pearson, 2018 (ISBN 978-1-292-17860-8).

Question 6.1

Using the Euler-Bernoulli equation show that a travelling flexural wave in a uniform beam will have a wave speed c that is not constant but depends on the wavelength. Assume a travelling wave deformation

$$w(x,t) = A_0 \sin\left[\frac{2\pi}{\lambda}(x - ct)\right]$$

and find how c depends on the wavelength λ. Is the result physically reasonable? Spoiler alert: it is not. What is the problem then?

Solution

$$c = \frac{2\pi}{\lambda}\sqrt{\frac{EI}{\rho A}}$$



Question 6.2

A uniform steel cantilever beam has the following dimensions: length $L = 0.4$ m, width $a = 35$ mm and depth $b = 8$ mm. Assume that $E = 207$ GPa and $\rho = 7840$ kg/m^3. Table 1 compares some natural frequency calculations (all values in Hz). Fill in the missing values.

Table 1: Frequencies calculated with different methods, in Hz (Question 6.2).

Method	Mode 1	Mode 2	Mode 3
Single mass model (try 1/3 of beam mass) at end of massless beam	?		
Rayleigh's method with parabolic assumed mode shape	?		
20 mass lumped mass model	41.45	259.06	723.53
FE using 4 elements	41.51	260.40	733.80
Euler-Bernoulli solution	?	?	?

Solution

Table 2: Frequencies calculated with different methods, in Hz (Question 6.2).

Method	Mode 1	Mode 2	Mode 3
Single mass model (try 1/3 of beam mass) at end of massless beam	35.57		
Rayleigh's method with parabolic assumed mode shape	53.03		
20 mass lumped mass model	41.45	259.06	723.53
FE using 4 elements	41.51	260.40	733.80
Euler-Bernoulli solution	41.69	263.33	731.50



Question 6.3

A uniform beam AB is simply supported at end A ($x = L$) while end B ($x = 0$) is rigidly clamped to a rigid wall. Apply these boundary conditions and show that the roots (kL) of the following equation

$$\tan(kL) = \tanh(kL)$$

can be used to determine the beam frequencies. Use a sketch plot to show where the roots are located. Determine the approximate value of the first root.

Solution

The approximate solution is $(kL) \approx \pi(n + 1/4)$.



Question 6.4

A long uniform beam is suspended from a frictionless pivot and hangs vertically. If disturbed the beam will behave like a pendulum. The frequency of the pendulum can be calculated assuming the beam is rigid. However, the beam is not rigid and, if struck sideways, will oscillate at many frequencies as well as swinging as a pendulum.
Use Euler-Bernoulli's equation to find a frequency equation for the pendulum beam. Assume that gravity has a negligible effect on this.

Solution

The approximate solution is $(kL) \approx \pi(n + 1/4)$. Considering the definition for k, the natural frequencies can be determined to be

$$\omega = \left[\frac{\pi}{L}\left(n + \frac{1}{4}\right)\right]^2 \sqrt{\frac{EI}{\rho A}}$$



Question 6.5

If a cantilever beam of length L has an additional mass M attached at the free end what boundary conditions would apply to the Euler-Bernoulli solution? Hint: the shear force at the beam end would not be zero, but would be related to the transverse inertia force of the mass moving with acceleration of the beam end.

Solution

The boundary conditions at $x = L$ are

$$\left.\frac{\partial^2 w}{\partial x^2}\right|_{x=L} = 0 \quad \text{and} \quad EI \left.\frac{\partial^3 w}{\partial x^3}\right|_{x=L} = m\ddot{w}(L)$$



Question 6.6

The structure shown in Figure 1 is a circular cross-section (radius r) beam/bar clamped at one end and free at the other. Assume a travelling wave approach and obtain the ratio between the transverse and longitudinal (axial) wave speeds for the beam/bar.

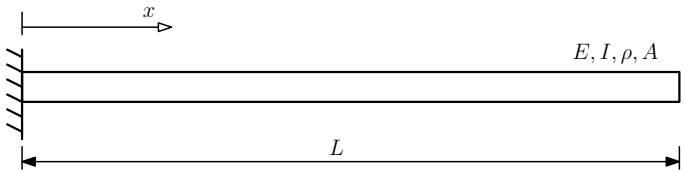


Figure 1: Clamped beam/bar for Question 6.6.

Solution

The ratio of the wave speeds is

$$\frac{c_\mathrm{t}}{c_\mathrm{l}} = \frac{\pi r}{\lambda_\mathrm{t}}$$

Question 6.7

The beams shown in Figures 2 (a) to (d) are clamped at $x = 0$ and have different boundary conditions at $x = L$. The beams have circular cross section ($r = 10$ mm), $L = 2$ m and are made from steel with $E = 200$ GPa and $\rho = 8000$ kg/m^3. Consider $K = 1.0 \times 10^6$ N/m and a standing wave solution.

a) State the boundary conditions for all the beams in figures 2(a) to (d).
b) Obtain the first three wave frequencies for the beams in figures 2(a) and (b).
c) Compare and discuss the obtained frequencies.

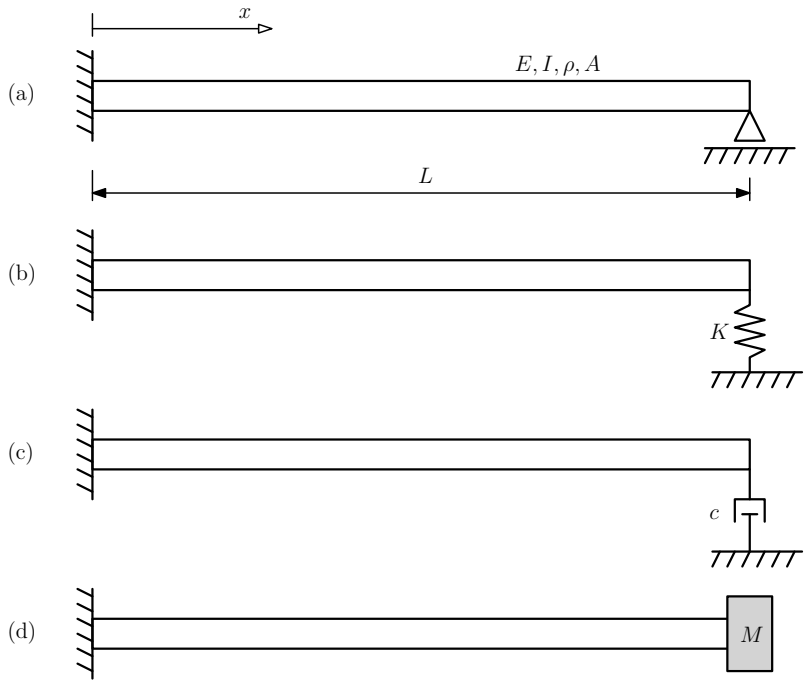


Figure 2: Clamped beams with different boundary conditions for Question 6.7.

Solution

Table 3: First three wave frequencies for the beams in Figures 2 (a) and (b) for Question 6.7.

	(a)			(b)		
n Frequency	kL	ω_n [rad]	ω_n [Hz]	kL (b)	ω_n [rad]	ω_n [Hz]
1	3.9266	96.36	15.34	3.9210	96.09	15.29
2	7.0686	312.28	49.70	7.0331	309.15	49.20
3	10.2102	651.55	103.70	10.0981	637.32	101.43



Question 6.8

A uniform elastic beam of length L, cross-section area A and second moment of area I has density ρ and elastic modulus E. Find the mode shapes and modal frequencies if the beam is simply supported.

Solution

The modal frequencies of the beam are

$$\omega = \left(\frac{\pi n}{L}\right)^2 \sqrt{\frac{EI}{A\rho}} \quad \text{for} \quad n = 1, 2, \ldots$$

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impact_on_structures

Dynamics 4

Exercise Sheet 7

Notes for this Exercise Sheet This exercise sheet covers the following topics: (i) free vibration of beams and flexural wave propagation in continuous uniform beams; (ii) beam shape functions and boundary conditions; (iii) Rayleigh-Ritz mode approximation; (iv) impact on slender structural elements.

Question 7.1

A uniform and homogeneous beam of length L, with Young's modulus E and cross-sectional second moment of area I has both ends fully clamped. This beam satisfies the equation of motion

$$0 = \frac{\partial^2 w}{\partial t^2} + \gamma^2 \frac{\partial^4 w}{\partial x^4} \quad \text{with} \quad \gamma^2 = \frac{EI}{\rho A}$$

a) What are the boundary conditions for the beam?
b) Suggest adequate shape functions $X_1(x)$ and $X_2(x)$ for the first (symmetric) and second (antisymmetric) mode shapes of the beam.
c) Estimate the frequencies of the first symmetric and antisymmetric modes.

Solution

The natural frequency for the first (symmetrical) mode shape of the beam is

$$\omega_1 = 22.8 \frac{\gamma}{L^2}$$

and the natural frequency of the second (antisymmetrical) mode shape is

$$\omega_2 = 78.9 \frac{\gamma}{L^2}$$



Question 7.2

A uniform beam of length L and mass $M = \rho AL$ is fully clamped at both ends. For the mode shape approximation

$$X(x) = \frac{1}{2}\left[1 - \cos\left(\frac{2\pi x}{L}\right)\right]$$

find the equivalent mass $\bar{M}$ and modal frequency ω for free vibration of the beam.

Solution

See solution of previous question.



Question 7.3

A uniform elastic beam of length L, area of cross-section A, elastic modulus E and second moment of area I is fully clamped at both ends. The beam is initially at rest before it is impacted at mid-span by a rigid missile of mass m, travelling with speed v_0 in a direction perpendicular to the axis of the beam, as shown in Figure 1.

a) Find the response of the beam for a mass ratio $m/M = 0.1$.

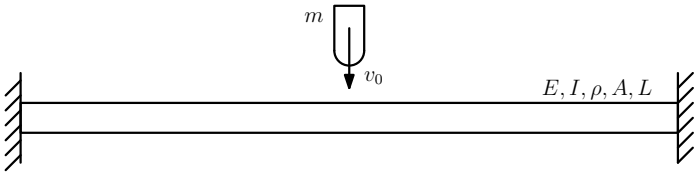


Figure 1: Impact of a rigid missile on a uniform beam (Question 7.3).

Solution

Table 1: Modified frequencies and amplitudes for the impacted beam in Figure 1.

m/M	$\bar{\omega}_1 L^2/\gamma$	$A_1\gamma/(L^2 v_0)$	$\bar{\omega}_2 L^2/\gamma$	$A_2\gamma/(L^2 v_0)$
0.1	21.4	0.006	79.0	0.0



Question 7.4

A uniform elastic beam of length L, area of cross-section A and second moment of area I has density ρ and elastic modulus E and is simply supported. The beam is at rest when it is impacted at mid-span by a rigid missile of mass m and velocity v_0 (see Figure 2). For shape functions

$$X_j(x) = \sin(k_j x) \quad \text{with} \quad k_j = \frac{j\pi}{L}$$

find the equivalent mass $\bar{M}_j$, the equivalent stiffness $\bar{K}_j$ and the amplitude A_j of the response of the beam as a function of the mass ratio $m/M = \beta$.

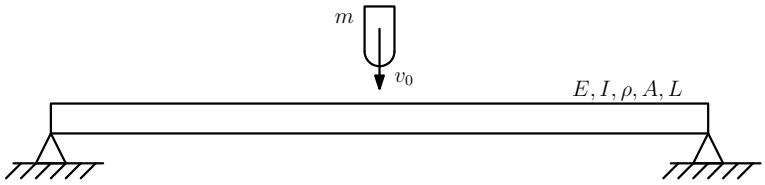


Figure 2: Impact of a rigid missile on a uniform simply supported beam (Question 7.4).

Solution

See solution of previous question.



Question 7.5

A uniform elastic beam of length L, area of cross-section A, elastic modulus E and second moment of area I is simply supported. The beam is initially at rest before it is impacted at $x = L/3$ by a rigid missile of mass m, travelling with speed v_0 in a direction perpendicular to the axis of the beam.

a) Find the response of the beam if the missile weighs 10% of the weight of the beam.
b) Find the response of the beam if the missile weighs 1% of the weight of the beam.

Solution

See solution of previous question, with $x_0 = L/3$ and $\beta = 0.1$:

$$\omega_1^2 = \left(\frac{\pi}{L}\right)^4 \frac{EI}{\rho A}, \qquad \text{and} \qquad \bar{\omega}_1^2 = \left(\frac{\pi}{L}\right)^4 \frac{EI}{\rho A} \Big/ \left[1 + 0.2 \sin\left(\frac{\pi}{3}\right)\right]$$

$$\omega_2^2 = \left(\frac{2\pi}{L}\right)^4 \frac{EI}{\rho A}, \qquad \text{and} \qquad \bar{\omega}_2^2 = \left(\frac{2\pi}{L}\right)^4 \frac{EI}{\rho A} \Big/ \left[1 + 0.2 \sin\left(\frac{2\pi}{3}\right)\right]$$

See solution of previous question, with $x_0 = L/3$ and $\beta = 0.01$:

$$\omega_1^2 = \left(\frac{\pi}{L}\right)^4 \frac{EI}{\rho A}, \qquad \text{and} \qquad \bar{\omega}_1^2 = \left(\frac{\pi}{L}\right)^4 \frac{EI}{\rho A} \Big/ \left[1 + 0.02 \sin\left(\frac{\pi}{3}\right)\right]$$

$$\omega_2^2 = \left(\frac{2\pi}{L}\right)^4 \frac{EI}{\rho A}, \qquad \text{and} \qquad \bar{\omega}_2^2 = \left(\frac{2\pi}{L}\right)^4 \frac{EI}{\rho A} \Big/ \left[1 + 0.02 \sin\left(\frac{2\pi}{3}\right)\right]$$



Question 7.6

The thin metal blade in Figure 3 is made of brass (elastic modulus $E = 97$ GPa, density $\rho = 8500$ kg/m^3) and used as a tuning device for musical instruments.

a) Find the thickness of the blade t that will ensure the blade has a natural frequency that's equal to a key of A_4, which is 440 Hz.
b) If the blade is made to vibrate by impacting it with a metal sphere of mass $m = 1$ g with speed $v_0 = 1.4$ m/s at $x_0 = 2L/3$, how much out of tune will the blade resonate? Give your answer in percentage relative to the key of A_4 (440 Hz).
c) Find the corresponding amplitude of vibration of the tuning blade.

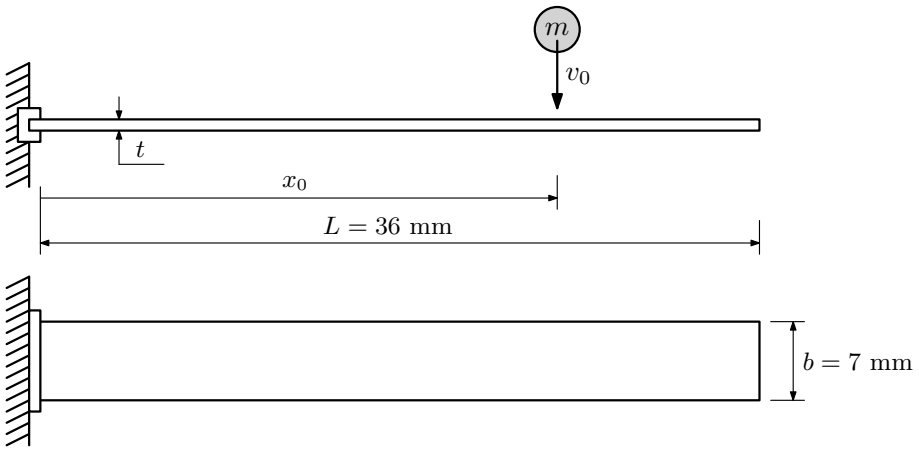


Figure 3: Tuning metallic blade for Question 7.6.

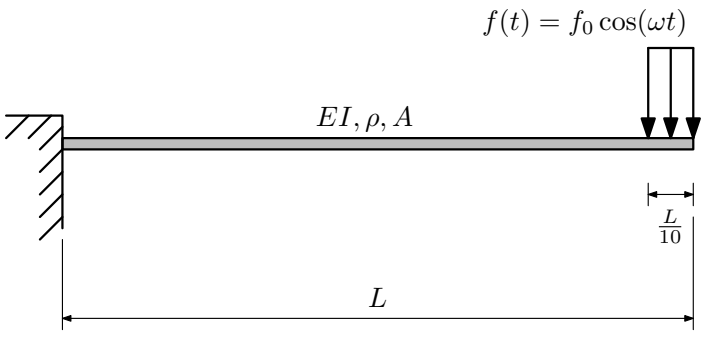
Solution

The amplitude of vibration of the tuning blade is $A_1 = 1.0$ mm.

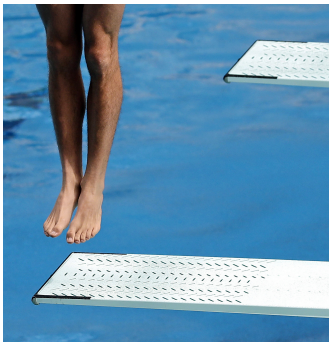
Question 7.7

The cantilever beam shown in Figure 4a is a model of the 3 meter Olympic diving board in Figure 4b[1]. The board has density ρ, cross section area A, length L and bending stiffness EI. A pulsating distributed force $f(t) = f_0 \cos(\omega t)$ acts on the board as shown.

a) Suggest two alternative normalised shape functions for the system.
b) Select one of the shape functions from part (a) and derive the equation of motion for the system.
c) Consider that $L = 3$ m, the width and thickness of the board are $b = 500$ mm and $h = 30$ mm, respectively, and determine if pine wood ($\rho = 500$ kg/m^3, $E = 9$ GPa) is a good material to ensure that the natural frequency of the board is lower than 10 rad/s.



(a)



(b)

Figure 4: Model of the 3 meter Olympic diving board for Question 7.7.

Solution

Pine wood is not a good material for the diving board (elastic modulus is not lower than 3 GPa).

[1]Photo licensed under the Creative Commons Attribution 2.0 Generic License (CC BY 2.0).



Question 7.8

Consider a simply-supported homogeneous beam of length L, density ρ, elastic modulus E, cross-section area A and second moment of area I. The structure is impacted by a projectile of mass m with velocity v_0 at location x_0 from one end of the beam.

a) Propose a trigonometric shape function for the first natural frequency of the beam.
b) Show that the first natural frequency is independent of the normalisation of the shape function.
c) Investigate if the modified frequency $\bar{\omega}$ and amplitude of vibration A depend on the normalisation of the shape function.

Solution

A possible trigonometric shape function for the first natural frequency of the beam is

$$\varphi(x) = \sin\left(\frac{\pi x}{L}\right)$$

The modified frequency $\bar{\omega}$ and amplitude of vibration A are only correct if the shape function is normalised, such as the one above.