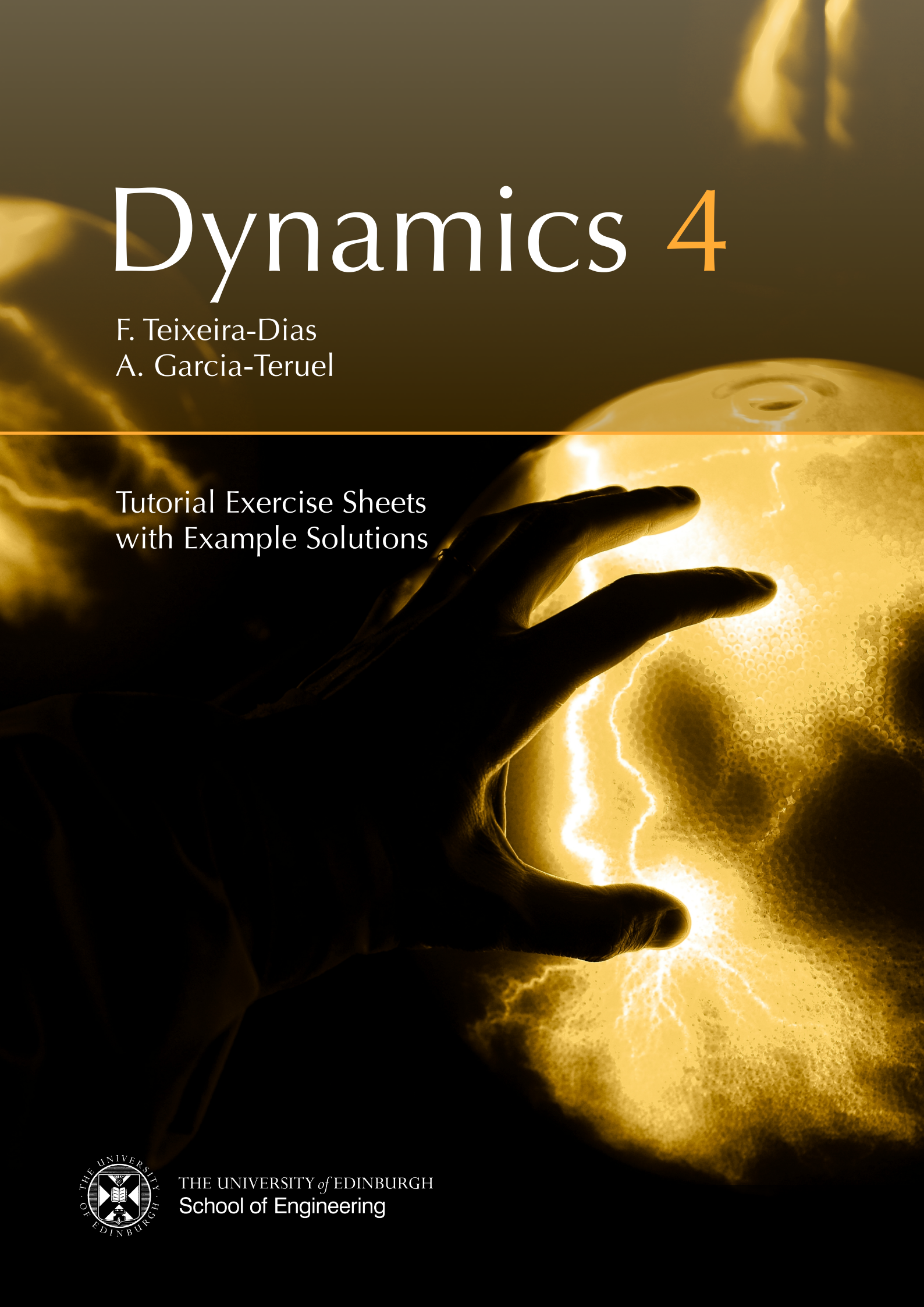
Dynamics 4

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Tutorial Exercise Sheets
with Example Solutions



THE UNIVERSITY of EDINBURGH
School of Engineering

Dynamics 4
MECE10002
Semester 1, 2026-27

Classroom
Classroom 12, Alrick

Learning module
LM1

Course notes
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Dynamics 4

Exercise Sheet 1

Notes for this Exercise Sheet The topics covered in this exercise sheet are: (i) deriving and understanding equations of motion of a dynamical system; (ii) Using Lagrange's equations as a straightforward approach to obtain the equations of motion; (iii) determine the kinetic and potential energy expressions for different types of dynamical systems; (iv) analyse dynamic systems with different characteristics (e.g. linear and rotation movements, extension and torsion elastic potential energy. etc.) All these topics were discussed in the lectures and are covered in the lecture notes. If you want to read more about this or go through more examples you can, for example, see Sections 6.6 and 6.7 (pages 619-624) on the reference suggested for further reading[a]. Additionally, you can solve problems 6.1 to 6.6 on pages 675 and 676 on this same reference.

[a] S.S. Rao, Mechanical Vibrations (6^{th} edition, SI), Pearson, 2018 (ISBN 978-1-292-17860-8).

Question 1.1

The system in Figure 1 is a pendulum with an imposed vertical motion of the pivot. This oscillatory motion is $\bar{z}(t) = A\cos(\omega t)$. Although strictly this is not a conservative system (the potential energy is time-dependent) it is a special case and the conservative form of Lagrange's equations can in fact be applied. Determine the system's differential equations of motion using Lagrange's method.

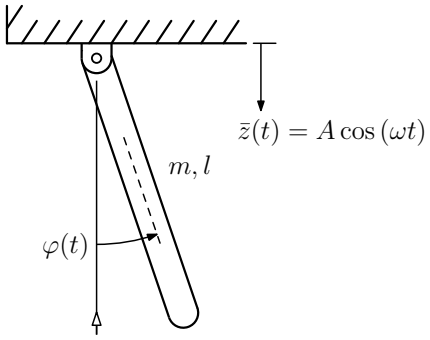


Figure 1: Dynamic system for Question 1.1

Example solution

We can start by determining the coordinates of the centre of gravity (CoG) of the bar and their respective derivatives, since these will be used for the kinetic and potential energy expressions:

$$x_{CG} = \frac{1}{2} l \sin(\varphi)$$
$$y_{CG} = -\left[\frac{1}{2} l \cos(\varphi) + \bar{z}(t)\right]$$

and

$$\dot{x}_{CG} = \frac{1}{2} l \cos(\varphi)\dot{\varphi}$$
$$\dot{y}_{CG} = \frac{1}{2} l \sin(\varphi)\dot{\varphi} - \dot{\bar{z}}(t)$$

The motion of the pendulum can be described through superposition of the rotation of the bar around its CoG and the linear motion of its mass. The total kinetic energy of the system is then

$$
\begin{aligned}
T &= \frac{1}{2}I_{CG}\dot{\varphi}^2 + \frac{1}{2}m\dot{x}_{CG}^2 + \frac{1}{2}m\dot{y}_{CG}^2 \\
&= \frac{1}{2}I_{CG}\dot{\varphi}^2 + \frac{1}{2}m\left(\frac{1}{2}l\cos\varphi\dot{\varphi}\right)^2 + \frac{1}{2}m\left(\frac{1}{2}l\sin\varphi\dot{\varphi} - \dot{\bar{z}}\right)^2 \\
&= \frac{1}{2}I_{CG}\dot{\varphi}^2 + \frac{1}{2}m\left(\frac{1}{4}l^2\dot{\varphi}^2\cos^2\varphi + \frac{1}{4}l^2\dot{\varphi}^2\sin^2\varphi + \dot{\bar{z}}^2 - l\dot{\varphi}\dot{\bar{z}}\sin\varphi\right) \\
&= \frac{1}{24}ml^2\dot{\varphi}^2 + \frac{1}{2}m\left(\frac{1}{4}l^2\dot{\varphi}^2 + \dot{\bar{z}}^2 - l\dot{\varphi}\dot{\bar{z}}\sin\varphi\right) \\
&= \frac{1}{6}ml^2\dot{\varphi}^2 + \frac{1}{2}m\left(\dot{\bar{z}}^2 - l\dot{\varphi}\dot{\bar{z}}\sin\varphi\right)
\end{aligned}
$$

Since the vertical oscillatory motion forced upon the system is known, $\bar{z}(t)$ is not a system coordinate, but will influence the potential energy of the system by changing the value of the y-coordinate of the CoG. Therefore, assuming $V = 0$ at $y = 0$, the potential energy of the system becomes

$$
\begin{aligned}
V &= mgy_{CG} \\
&= -mg\left(\frac{1}{2}l\cos\varphi + \bar{z}\right)
\end{aligned}
$$

This can again be considered a conservative system, so that the Lagrange equations will equal zero,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0$$

leading to

$$
\begin{aligned}
\frac{\partial T}{\partial\dot{\varphi}} &= \frac{2}{6}ml^2\dot{\varphi} - \frac{1}{2}ml\dot{\bar{z}}\sin\varphi \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{\varphi}}\right) &= \frac{1}{3}ml^2\ddot{\varphi} - \frac{1}{2}ml\dot{\varphi}\dot{\bar{z}}\cos\varphi - \frac{1}{2}ml\ddot{\bar{z}}\sin\varphi \\
\frac{\partial T}{\partial\varphi} &= -\frac{1}{2}ml\dot{\varphi}\dot{\bar{z}}\cos\varphi, \qquad \frac{\partial V}{\partial\varphi} = \frac{1}{2}mgl\sin\varphi
\end{aligned}
$$

Inserting these in the Lagrange equation leads to

$$
\begin{aligned}
&\frac{ml^2}{3}\ddot{\varphi} + \frac{ml}{2}\left(\dot{\varphi}\dot{\bar{z}}\cos\varphi - \ddot{\bar{z}}\sin\varphi - \cos\varphi\dot{\varphi}\dot{\bar{z}} + g\sin\varphi\right) = 0 \\
&\frac{ml^2}{3}\ddot{\varphi} - \frac{ml}{2}\ddot{\bar{z}}\sin\varphi + \frac{mgl}{2}\sin\varphi = 0
\end{aligned}
$$

The linearised equation of motion for this system is then

$$\frac{ml^2}{3}\ddot{\varphi} - \frac{ml}{2}\ddot{\bar{z}}\varphi + \frac{mgl}{2}\varphi = 0$$

Question 1.2

Consider the dynamic system shown in Figure 2. AB is a rigid uniform bar of mass M, pivoted at point A, with an added attached mass m as shown. Assume the angular deflection of AB is small and neglect the gravitational potential energy.

a) Determine the system's kinetic and potential energy expressions.
b) Determine the corresponding equations of free vibration (equations of motion).
c) Explain in what circumstances it is a reasonable assumption to neglect gravitational potential energy in this case.

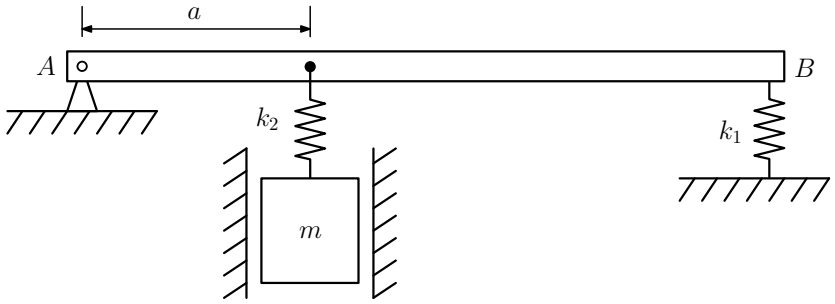


Figure 2: Dynamic system for Question 1.2

Example solution

Neglecting gravitational potential energy and considering that $\theta(t)$ is the rotation of the bar and $x(t)$ is the vertical displacement of the mass, we obtain the energy functions. The total kinetic energy of the system is

$$T = \frac{1}{2} I_A \dot{\theta}^2 + \frac{1}{2} m \dot{x}^2$$

where $I_A = \frac{1}{3} M L^2$ is the inertia of bar AB about point A. The potential energy of the system, ignoring gravitational potential energy, is

$$V = \frac{1}{2} k_1 \left[L \sin\theta \right]^2 + \frac{1}{2} k_2 \left[x - a \sin\theta \right]^2$$

with

$$(x - a\sin\theta)^2 = x^2 - 2xa\sin\theta + a^2 \sin^2\theta$$

For a conservative system — with no external forces or dissipative terms— the Lagrange equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0$$

In the current system, $q_1 = x(t)$ and $q_2 = \theta(t)$. Applying Lagrange's method to each generalised coordinate leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} + \frac{\partial V}{\partial x} = 0 \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} = 0$$

For the generalised coordinate $x(t)$ we obtain

$$\frac{\partial T}{\partial \dot{x}} = m\dot{x}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{x}}\right) = m\ddot{x}$$
$$\frac{\partial T}{\partial x} = 0, \qquad \frac{\partial V}{\partial x} = k_2(x - a\sin\theta)$$

We insert these in the Lagrange equation to obtain the equation of motion for the generalised displacement $x(t)$,

$$m\ddot{x} + k_2(x - a\sin\theta) = 0$$
$$m\ddot{x} + k_2 x - k_2 a \sin\theta = 0$$

which can be linearised with $\sin\theta \approx \theta$, leading to

$$m\ddot{x} + k_2 x - k_2 a\theta = 0$$



For the generalised coordinate $\theta(t)$ we obtain

$$\frac{\partial T}{\partial \dot{\theta}} = I_A \dot{\theta}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = I_A \ddot{\theta}, \qquad \frac{\partial T}{\partial \theta} = 0$$
$$\frac{\partial V}{\partial \theta} = k_1 L^2 \sin\theta\cos\theta - k_2 x a \cos\theta + k_2 a^2 \sin\theta\cos\theta$$

Inserting these in the corresponding Lagrange equation we obtain the equation of motion for $\theta(t)$

$$I_A\ddot{\theta} + k_1 L^2 \sin\theta\cos\theta - k_2 x a \cos\theta + k_2 a^2 \sin\theta\cos\theta = 0$$

which can be linearised with $\sin\theta \approx \theta$ and $\cos\theta \approx 1$

$$I_A\ddot{\theta} + (k_1 L^2 + k_2 a^2)\theta - k_2 a x = 0$$

The equations of motion for this system become:

$$m\ddot{x} - k_2 a\theta + k_2 x = 0$$
$$I_A\ddot{\theta} + \left(k_1 L^2 + k_2 a^2\right)\theta - k_2 a x = 0$$

In this case it is reasonable to ignore gravitational potential energy due to the considered small vibrations, which will result in a very small variation of gravitational potential energy compared to the potential energy stored in the two elastic springs.

Question 1.3

The mass M in the system shown in Figure 3 is restrained by springs and has a pendulum bar hanging from it. Neglect friction and determine the energy expressions and the system's differential equations of motion. Show that the motion of the mass and the angular motion of the pendulum are coupled (through non-linear terms). Identify these non-linear terms and obtain the linearised equations of motion of the system.

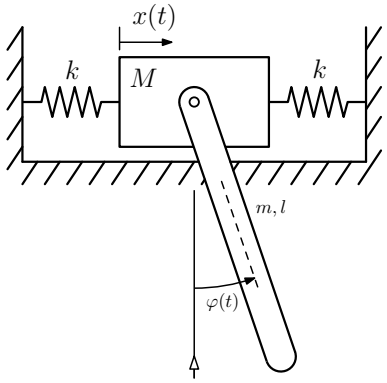


Figure 3: Dynamic system for Question 1.3

Example solution

The system has two degrees of freedom: $x(t)$, the linear displacement of mass M, and $\varphi(t)$, the rotation angle of the bar. The kinetic energy of the system can be divided in two main contributions: the linear kinetic energy of mass M, which is

$$T_M = \frac{1}{2} M \dot{x}^2$$

and the rotation (angular) kinetic energy of the bar, which can be separated into the rotation of the bar about its centre of mass (CG) and the linear motion of the centre of mass (horizontal and vertical). This leads to

$$T_{\text{bar}} = \frac{1}{2} m \dot{x}_{\text{CG}}^2 + \frac{1}{2} m \dot{y}_{\text{CG}}^2 + \frac{1}{2} I_{\text{CG}} \dot{\varphi}^2$$

where $I_{\text{CG}} = ml^2/12$ is the inertia of the bar about its centre of mass. The coordinates of the centre of mass of the bar are

$$x_{\text{CG}} = x(t) + \frac{l}{2}\sin\varphi \qquad \text{and} \qquad y_{\text{CG}} = -\frac{l}{2}\cos\varphi$$

and the corresponding velocities

$$\dot{x}_{\text{CG}} = \dot{x}(t) + \dot{\varphi}\frac{l}{2}\cos\varphi \qquad \text{and} \qquad \dot{y}_{\text{CG}} = \dot{\varphi}\frac{l}{2}\sin\varphi$$

The kinetic energy of the whole system is then

$$T = T_M + T_{\text{bar}} = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m \dot{x}_{\text{CG}}^2 + \frac{1}{2} m \dot{y}_{\text{CG}}^2 + \frac{1}{2} I_{\text{CG}} \dot{\varphi}^2$$

Substituting the velocities of the centre of mass and inertia of the bar yields

$$\begin{aligned} T &= \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m \left[\left(\dot{x} + \dot{\varphi}\frac{l}{2}\cos\varphi \right)^2 + \dot{\varphi}^2 \frac{l^2}{4}\sin^2\varphi \right] + \frac{1}{2}\frac{ml^2}{12}\dot{\varphi}^2 \\ &= \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m \left(\dot{x}^2 + \dot{\varphi}^2 \frac{l^2}{4} + \dot{x}\dot{\varphi} l \cos\varphi \right) + \frac{1}{24} m l^2 \dot{\varphi}^2 \\ &= \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{6} m l^2 \dot{\varphi}^2 + \frac{1}{2} m l \dot{x}\dot{\varphi}\cos\varphi \end{aligned}$$

The potential energy of the system can also be divided in two main contributions: the elastic potential energy of the two springs, which is

$$V_{\text{s}} = 2\left(\frac{1}{2} k x^2\right)$$

and the gravitational potential energy

$$V_{\mathrm{g}} = mgy_{\mathrm{CG}} = -mg\frac{l}{2}\cos\varphi$$

The total potential energy is then

$$V = V_{\mathrm{s}} + V_{\mathrm{g}} = kx^2 - mg\frac{l}{2}\cos\varphi$$

As this is a conservative system, Lagrange's equations can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0$$

where $q_1 = x(t)$ and $q_2 = \varphi(t)$ are the generalised coordinates of the system. Applying Lagrange's method to each of the generalised coordinates leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} + \frac{\partial V}{\partial x} = 0 \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\varphi}}\right) - \frac{\partial T}{\partial \varphi} + \frac{\partial V}{\partial \varphi} = 0$$

For the coordinate $x(t)$

$$\begin{aligned}
\frac{\partial T}{\partial \dot{x}} &= (M+m)\dot{x} + \frac{1}{2}ml\dot{\varphi}\cos\varphi \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{x}}\right) &= (M+m)\ddot{x} + \frac{1}{2}ml\ddot{\varphi}\cos\varphi - \frac{1}{2}ml\dot{\varphi}^2\sin\varphi \\
\frac{\partial T}{\partial x} &= 0 \\
\frac{\partial V}{\partial x} &= 2kx
\end{aligned}$$

Adding all the energy terms yields

$$(M+m)\ddot{x} + \frac{1}{2}ml\ddot{\varphi}\cos\varphi - \frac{1}{2}ml\dot{\varphi}^2\sin\varphi + 2kx = 0$$

Assuming small vibrations for $x(t)$ the previous equation of motion can be linearised considering that higher order terms are negligible, $\cos\varphi \approx 1$ and $\sin\varphi \approx \varphi$, leading to

$$(M+m)\ddot{x} + \frac{1}{2}ml\ddot{\varphi} + 2kx = 0$$

For the coordinate $\varphi(t)$

$$\begin{aligned}
\frac{\partial T}{\partial \dot{\varphi}} &= \frac{1}{3}ml^2\dot{\varphi} + \frac{1}{2}ml\dot{x}\cos\varphi \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\varphi}}\right) &= \frac{1}{3}ml^2\ddot{\varphi} + \frac{1}{2}ml\ddot{x}\cos\varphi - \frac{1}{2}ml\dot{x}\dot{\varphi}\sin\varphi \\
\frac{\partial T}{\partial \varphi} &= -\frac{1}{2}ml\dot{x}\dot{\varphi}\sin\varphi \\
\frac{\partial V}{\partial \varphi} &= \frac{1}{2}mgl\sin\varphi
\end{aligned}$$

Adding all the energy terms yields

$$\frac{1}{3}ml^2\ddot{\varphi} + \frac{1}{2}ml\ddot{x}\cos\varphi - \frac{1}{2}ml\dot{x}\dot{\varphi}\sin\varphi + \frac{1}{2}ml\dot{x}\dot{\varphi}\sin\varphi + \frac{1}{2}mgl\sin\varphi = 0$$

Assuming small vibrations for $\varphi(t)$ the previous equation of motion can be linearised considering that higher order terms are negligible, $\cos\varphi \approx 1$ and $\sin\varphi \approx \varphi$, leading to

$$\frac{1}{3}ml^2\ddot{\varphi} + \frac{1}{2}ml\ddot{x} + \frac{1}{2}mgl\varphi = 0$$

The linearised equations of motion for this system are then

$$\begin{aligned}
(M+m)\ddot{x} + \frac{ml}{2}\ddot{\varphi} + 2kx &= 0 \\
\frac{ml^2}{3}\ddot{\varphi} + \frac{ml}{2}\ddot{x} + \frac{mgl}{2}\varphi &= 0
\end{aligned}$$

Question 1.4

Figure 4 shows a torsional vibration system. A motor drives a centrifugal pump impeller through a reduction gear. $I_1 = 25$ kgm^2, $I_2 = 15$ kgm^2 and $I_3 = 35$ kgm^2 are the inertias of the motor armature, the gear wheel and the combined gear wheel and pump rotor, respectively. The gear ratio is $3.5:1$ ($\beta = 3.5$). The power transmission shaft has a torsional stiffness $K_\mathrm{s} = 20\times10^6$ Nm (Nm/rad). The pump has 8 blades and runs at 5250 rpm.

a) Use Lagrange's method to get the system's differential equations for torsional vibration, i.e. its equations of motion.

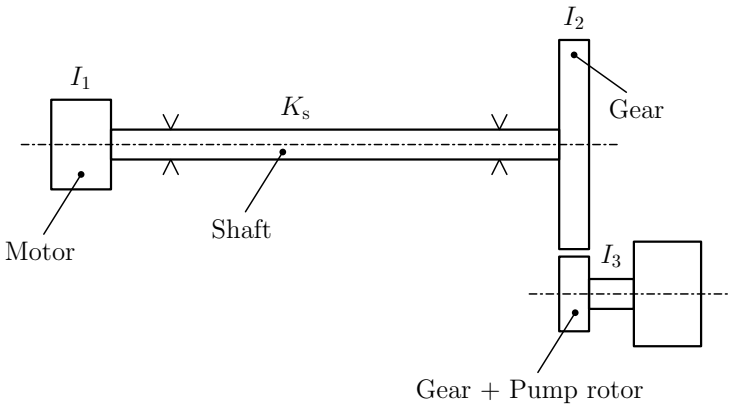


Figure 4: Dynamic system for Question 1.4

Example solution

We start by noting that θ_3 and θ_2 are kinematically linked through the gear ratio, that is $\theta_3 = \beta\theta_2$ where $\beta = 3.5$, leading to the following relation between rotational (angular) velocities:

$$\dot{\theta}_3 = \beta\dot{\theta}_2$$

The system has two generalised coordinates (degrees-of-freedom). One is θ_1 and the other can be either θ_2 or θ_3 (we will be using θ_2 in this example solution). The kinetic energy of the whole system is:

$$\begin{aligned} T &= \frac{1}{2}I_1\dot{\theta}_1^2 + \frac{1}{2}I_2\dot{\theta}_2^2 + \frac{1}{2}I_3\dot{\theta}_3^2 \\ &= \frac{1}{2}I_1\dot{\theta}_1^2 + \frac{1}{2}\left(I_2 + \beta^2 I_3\right)\dot{\theta}_2^2 \end{aligned}$$

and the potential energy of the system is:

$$V = \frac{1}{2}K_\mathrm{s}\left(\theta_2 - \theta_1\right)^2$$

where $(\theta_2 - \theta_1)$ is the shaft twist. Using T and V and applying Lagrange's method leads to the following equations of motion for the system:

$$\begin{cases} I_1\ddot{\theta}_1 + K_\mathrm{s}\theta_1 - K_\mathrm{s}\theta_2 = 0 \\ \left(I_2 + \beta^2 I_3\right)\ddot{\theta}_2 - K_\mathrm{s}\theta_1 + K_\mathrm{s}\theta_2 = 0 \end{cases}$$

These are the coupled ordinary differential equations (ODE) of motion of the system.

Dynamics 4

Exercise Sheet 2

Notes for this Exercise Sheet The topics covered in this exercise sheet are: (i) derive and understand equations of motion of dynamical systems; (ii) use Lagrange's equations to obtain the equations of motion for non-conservative dynamic systems; (iii) determine generalised forces for non-conservative systems. All these topics were discussed in the lectures and are covered in the lecture notes. If you want to read more about this or go through more examples you can, for example, see Sections 6.6 and 6.7 (pages 619-624) on the reference suggested for further reading[a]. Additionally, you can solve example 6.18 (page 656) and problems 6.1 to 6.6 on pages 675 and 676 of this same reference.

[a] S.S. Rao, Mechanical Vibrations (6^{th} edition, SI), Pearson, 2018 (ISBN 978-1-292-17860-8).

Dynamics 4
MECE10002
Semester 1, 2026-27

Classroom
Classroom 12, Alrick

Learning module
LM2

Course notes
Section 2.3

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analytical_dynamics

Question 2.1

The system in Figure 1 is very similar to the one on Question 1.4. However, there is a damper and a constant force P applied to the end of the pendulum so that it is always acting along the centre line of the pendulum[1].

a) Find any generalised forces that need to be considered.
b) Determine the equations of motion of the system (e.g. by comparison with the solution from Question 1.4).

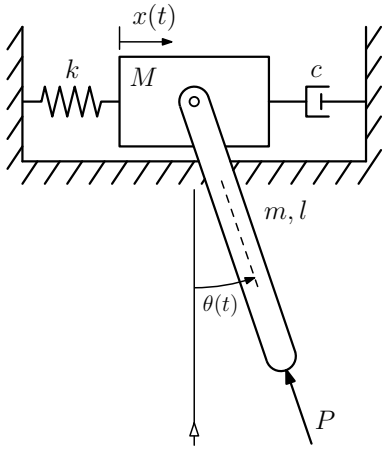


Figure 1: Dynamic system for Question 2.1.

Example solution

The solution of this question is very similar to that of Question 1.3 (see Exercise Sheet 1). The only differences are: (i) this system has one spring of stiffness k instead of two, (ii) it has a damper with damping constant c, and (iii) there is an external force P acting directly on the distributed mass pendulum, always aligned with its longitudinal axis.
The generalised force due to the damper is $-c\dot{x}$ and the generalised force due to the external force is $-P\sin\theta$. Note that both are acting on mass M because P is always aligned with the centre line of the pendulum. Consequently, the equations of motion of the system will be similar, provided these differences are accounted for, that is,

$$(M+m)\ddot{x} + \frac{1}{2}ml\ddot{\theta}\cos\theta - \frac{1}{2}ml\dot{\theta}^2\sin\theta + kx = -c\dot{x} - P\sin\theta$$

$$\frac{1}{3}ml^2\ddot{\theta} + \frac{1}{2}ml\ddot{x}\cos\theta + \frac{1}{2}mgl\sin\theta = 0$$

[1] This is termed a *follower force* and could, for example, represent the thrust force due to a jet of fluid flowing out of a nozzle on the pendulum.



Assuming small displacements and rotations, these equations of motion can be linearised, leading to

$$(M+m)\ddot{x} + \frac{1}{2}ml\ddot{\theta} + kx = -c\dot{x} - P\theta$$

$$\frac{1}{3}ml^2\ddot{\theta} + \frac{1}{2}ml\ddot{x} + \frac{1}{2}mgl\theta = 0$$

Question 2.2

Consider the dynamic system shown in Figure 2, similar to the one in Question 1.1. AB is a rigid uniform bar, pivoted at point A, with an added attached mass m as shown. Now there is a dynamic disturbing force $P(t) = P_0 \cos(\omega t)$ acting on the bar and a damper attached at the end (point B). Once again assume the angular deflection of AB is small and can thus be neglected. Ignore the gravitational potential energy and complete the equations of motion from Question 1.1 with the generalised forces.

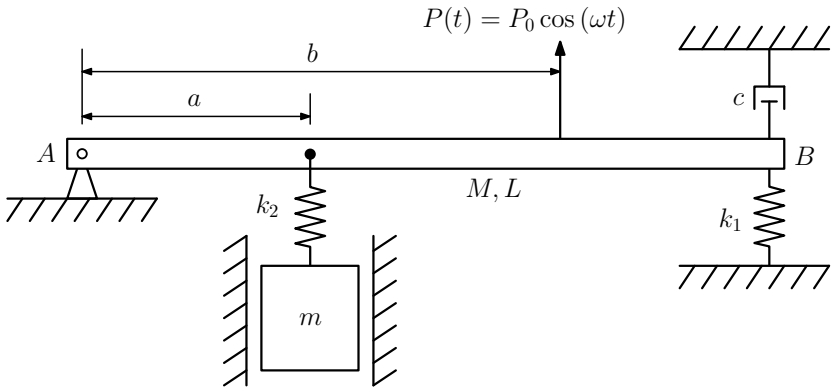


Figure 2: Dynamic system for Question 2.2.

Example solution

Observe that the bar AB does fixed axis rotation about point A while mass m moves vertically. Thus, this system has two degrees-of-freedom: (i) the rotation of the bar, $\theta(t)$, and (ii) the displacement of the mass, $y(t)$. The kinetic energy of the system is then

$$T = \frac{1}{2} I_A \dot{\theta}^2 + \frac{1}{2} m \dot{y}^2$$

where I_A is the inertia of the bar about point A, that is, $I_A = \frac{1}{3} M L^2$. The potential energy of the whole system is

$$V = \frac{1}{2} k_1 (L \sin\theta)^2 + \frac{1}{2} k_2 (y - a \sin\theta)^2 - Mg \frac{L}{2} \sin\theta - mgy$$

However, the two last terms — the gravity terms — only lead to steady weight forces on the system and do not affect its actual dynamic behaviour. Additionally, assuming the rotations of bar AB and vertical displacements of mass m are small, then these gravitational terms can be neglected, that is,

$$V = \frac{1}{2} k_1 L^2 \sin^2\theta + \frac{1}{2} k_2 (y - a \sin\theta)^2$$

Lagrange's equations for the system are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{y}}\right) - \frac{\partial T}{\partial y} + \frac{\partial V}{\partial y} = Q_y$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} = Q_\theta$$

where Q_y and Q_θ are the generalised forces associated to the $y(t)$ and $\theta(t)$ degrees-of-freedom, respectively. Differentiating the energy expressions leads to

$$\frac{\partial T}{\partial \dot{\theta}} = I_A \dot{\theta}\,; \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = I_A \ddot{\theta}\,; \quad \frac{\partial T}{\partial \theta} = 0$$

$$\frac{\partial T}{\partial \dot{y}} = m \dot{y}\,; \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{y}}\right) = m \ddot{y}\,; \quad \frac{\partial T}{\partial y} = 0$$

$$\frac{\partial V}{\partial \theta} = k_1 L^2 \sin\theta \cos\theta + k_2 a^2 \sin\theta \cos\theta - k_2 a y \cos\theta$$

$$\frac{\partial V}{\partial y} = k_2 y - k_2 a \sin\theta$$

Consequently, the equations of motion for this system are:

$$m\ddot{y} + k_2 y - k_2 a \sin\theta = Q_y$$
$$I_A\ddot{\theta} + \left(k_1 L^2 + k_2 a^2\right)\sin\theta\cos\theta - k_2 a y \cos\theta = Q_\theta$$

As we're assuming small rotations and displacements, these equations of motion can now be linearised, leading to,

$$m\ddot{y} + k_2 y - k_2 a\theta = Q_y$$
$$I_A\ddot{\theta} + \left(k_1 L^2 + k_2 a^2\right)\theta - k_2 a y = Q_\theta$$

The generalised forces Q_y and Q_θ can be evaluated from the virtual work done by the non-conservative forces acting on the system, that is,

$$\delta W_y = Q_y \delta y \qquad \text{and} \qquad \delta W_\theta = Q_\theta \delta\theta$$

So, in order to determine Q_y let δy be a virtual displacement in y (no change in θ). As there are no damper forces or dynamic loads on y then $Q_y = 0$. To find Q_θ we start by noting that the damper force is $-cL\dot{\theta}$ and assume a vertical generalised displacement $\delta\theta$ of the bar. Thus, the virtual work in $\delta\theta$ is

$$\delta w_\theta = -cL\dot{\theta} \times L\delta\theta + P_o\cos(\omega t) \times b\delta\theta$$

where b is the distance from point A to the force $P(t)$. Hence

$$Q_\theta = -cL^2\dot{\theta} + bP_o\cos(\omega t)$$

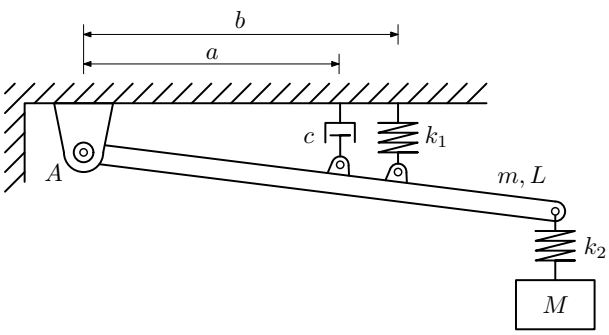
and the equations of motion for this system become

$$m\ddot{y} - k_2 a\theta + k_2 y = 0$$
$$I_A\ddot{\theta} + \left(k_1 L^2 + k_2 a^2\right)\theta - k_2 a y = -cL^2\dot{\theta} + bP_0\cos(\omega t)$$

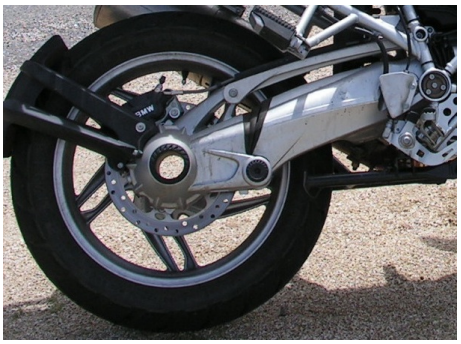
Question 2.3

The dynamic system shown in Figure 3a is a conceptual model of the motorbike suspension in Figure 3b[2]. The suspension arm has length L and mass m, and is pivoted at point A. The damping constant and the stiffness of the suspension are c and k_1, respectively. The mass of the wheel is M and the stiffness of the tyre is k_2. Neglect friction at the pivot point.

a) Identify the most appropriate generalised coordinates of the system and derive the corresponding kinetic and potential energy functions.
b) Find the generalised forces acting on the system.
c) Obtain the equations of motion of the system and linearise them if appropriate.



(a)



(b)

Figure 3: Motorbike suspension for Question 2.3.

Example solution

The system shown has two degrees-of-freedom: $y(t)$, the vertical displacement of the wheel represented by mass M, and $\theta(t)$, the rotation of the suspension arm about its pivot point A. The kinetic energy of the system has two components. One due to the vertical motion of the wheel (mass M) and the other due to the oscillation of the suspension arm. These terms are

$$T = \frac{1}{2}M\dot{y}^2 + \frac{1}{2}I_A\dot{\theta}^2$$

where the inertia of the arm about point A is

$$I_A = \frac{1}{3}mL^2$$

The potential energy of the system has two elastic components, associated to the stiffness of both springs, and two gravitational terms, associated to the vertical displacement of mass M and to the oscillation of the bar. Thus, the expression for the total potential energy of the suspension system can be written as

$$V = \frac{1}{2}k_1\left(b\sin\theta\right)^2 + \frac{1}{2}k_2\left(y - L\sin\theta\right)^2 - Mgy - mg\frac{L}{2}\sin\theta$$

The only relevant generalised force acting on the system is caused by the viscous damper and the oscillation of the suspension arm. This generalised force is

$$Q_\theta = -ca^2\frac{\mathrm{d}}{\mathrm{d}t}(\sin\theta) = -ca^2\dot{\theta}\cos\theta$$

As there are no external or dissipative (non-conservative) forces acting on the wheel (mass M), the generalised force associated with the $y(t)$ degree-of-freedom is $Q_y = 0$. Applying Lagrange's equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{y}}\right) - \frac{\partial T}{\partial y} + \frac{\partial V}{\partial y} = Q_y$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} = Q_\theta$$

[2]Photo licensed under the Creative Commons Attribution 2.0 Generic License (CC BY 2.0).



to the kinetic and potential energy terms, and to both degrees-of-freedom, the equation of motion for the $y(t)$ degree-of-freedom becomes

$$M\ddot{y} + k_2(y - L\sin\theta) - Mg = 0$$

and for the $\theta(t)$ degree-of-freedom,

$$\frac{1}{3}mL^2\ddot{\theta} - \frac{1}{2}mgL\cos\theta + k_1b^2\sin\theta\cos\theta - k_2L\cos\theta(y - L\sin\theta) = -ca^2\dot{\theta}\cos\theta$$

For the system shown it seems reasonable to assume both small angular motions (small θ) and small oscillations of the mass M (small y). As a consequence, both equations of motion can be linearised considering $\sin\theta \approx \theta$ and $\cos\theta \approx 1$, leading to

$$M\ddot{y} + k_2y - k_2L\theta - Mg = 0$$

and

$$\frac{1}{3}mL^2\ddot{\theta} - \frac{1}{2}mgL + k_1b^2\theta - k_2Ly + k_2L^2\theta = -ca^2\dot{\theta}$$

Question 2.4

You can use Lagrange's approach to get the equations of motion of a mass particle orbiting the Earth (e.g. a satellite). Start with the representation in figure 4, a moving mass m in free space with its position given by polar coordinates r and θ, both time functions.

a) Assume generic radial and tangential forces F_r and F_t, respectively, and use Lagrange's method to determine the equations of motion in r and θ.
b) Adapt these equations for a satellite orbiting outside the Earth's atmosphere. Assume $F_t = 0$ (no drag) and $F_r = -GMm/r^2$, that is, a radial inward force obeying Newton's inverse square law of gravitation. $M = 5.97 \times 10^{24}$ kg is the mass of the Earth and $G = 6.674 \times 10^{-11}$ N/(m/kg)^2 is the Universal Gravitational constant.

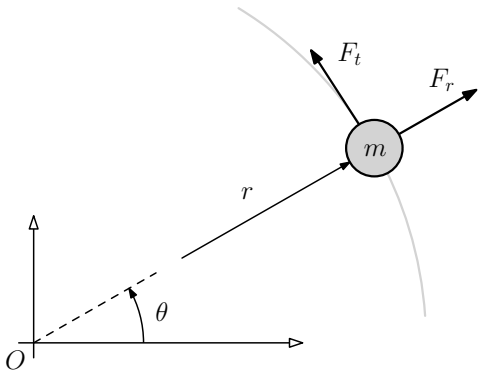


Figure 4: Orbiting satellite representation (Question 2.4).

Example solution

This question can be approached in two different ways: considering either (i) an orthogonal Oxy coordinate system or (ii) polar coordinates $Or\theta$. Using the polar coordinate system, F_r and F_t are the force components acting on mass m in the radial and tangential directions, respectively. Thus, the corresponding components of the velocity of m are

$$v_r = \dot{r} \qquad \text{and} \qquad v_\theta = r\dot{\theta}$$

respectively. The kinetic energy of the system can then be described by

$$T = \frac{1}{2}mv_r^2 + \frac{1}{2}mv_\theta^2 = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

Assuming the changes in r are small, it is possible to neglect gravitational potential energy, that is, $V = 0$. It should, however, be noted that the gravitational force has two distinct actions in this system: (i) the gravitational potential energy and (ii) it ensures the trajectory is elliptic (i.e. not a straight line). As such, and although the gravitational energy can be neglected because changes in r are small, the gravitational force has to be included as a generalised force, otherwise the satellite would have a linear trajectory.
Applying Lagrange's method and differentiating the kinetic and potential energy expressions leads to

$$\frac{\partial T}{\partial \dot{\theta}} = mr^2\dot{\theta}\,; \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta}\,; \quad \frac{\partial T}{\partial \theta} = 0$$

$$\frac{\partial T}{\partial \dot{r}} = m\dot{r}\,; \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{r}}\right) = m\ddot{r}\,; \quad \frac{\partial T}{\partial r} = mr\dot{\theta}^2$$

$$\frac{\partial V}{\partial \theta} = 0\,; \quad \frac{\partial V}{\partial r} = 0$$

The generalised forces acting on the system are

$$Q_r = F_r \qquad \text{and} \qquad Q_\theta = rF_t$$

Adding all the relevant terms, the equations of motion for this system become

$$m\ddot{r} - mr\dot{\theta}^2 = F_r$$
$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = rF_t$$



Now we apply central force motion to the satellite orbit case, where

$$F_r = -\frac{GMm}{r^2} \qquad \text{and} \qquad F_t = 0$$

with M being the mass of the planet. Thus, the equations of motion become

$$m\ddot{r} - mr\dot{\theta}^2 = -\tfrac{GMm}{r^2}$$
$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = 0$$

Dividing both equations by m (a non zero constant) leads to

$$\ddot{r} - r\dot{\theta}^2 = -\tfrac{GM}{r^2}$$
$$r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} = 0$$

showing that satellite motion is independent of the mass of the satellite, m. Solutions to these equations of motion can be periodic (i.e. ellipses, with circles as a special case) or non-periodic going off to infinity (i.e. hyperbolae)[3].

[3] See textbooks such as Beer and Johnston, *Vector mechanics for engineers* (3rd edition), Chapter 12, for a longer discussion on this topic.

Question 2.5

A bungee jumper wishes to try something new (!). He wants to start with the bungee cord strung out horizontally from an attachment and swing down in an arc. Obviously the cord is elastic and will extend during the swing. Consider the dynamic model for this system shown in figure 5. AB is a uniform rigid bar pivoted at A and with negligible mass. Mass m represents the jumper who slides on the bar (no friction) supported from A by the bungee cord (spring stiffness K and initial length a). Obtain the equations of motion of the system considering it is released from the horizontal position at time $t = 0$.

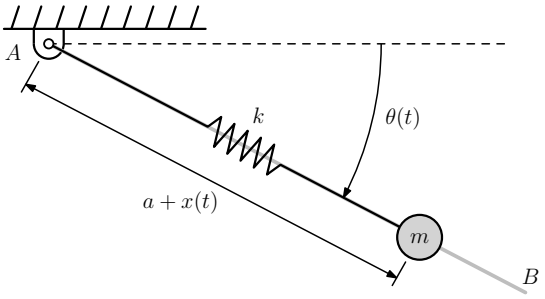


Figure 5: Dynamic model of a bungee jumper, for Question 2.5.

Example solution

Let us start by considering that a is the initial length of the spring (bungee cord) and that at the start $\theta = 0$ and $x = 0$. After time t the mass has a radial velocity $\dot{x}$ which is the velocity of m relative to the bar. The tangential component of the velocity is $(a + x)\dot{\theta}$. Consequently, the kinetic energy of the system is

$$T = \frac{1}{2}m\left[\dot{x}^2 + (a+x)^2\dot{\theta}^2\right]$$

The potential energy of the system is

$$V = \frac{1}{2}kx^2 - mg(a+x)\sin\theta$$

As there is no drag, there is no generalised force on the system and Lagrange's equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} + \frac{\partial V}{\partial x} = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} = 0$$

Differentiating the energy expressions leads to

$$\frac{\partial T}{\partial \dot{\theta}} = m(a+x)^2\dot{\theta}\ ; \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = m(a+x)^2\ddot{\theta} + 2m(a+x)\dot{x}\dot{\theta}\ ; \quad \frac{\partial T}{\partial \theta} = 0$$

$$\frac{\partial T}{\partial \dot{x}} = m\dot{x}\ ; \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{x}}\right) = m\ddot{x}\ ; \quad \frac{\partial T}{\partial x} = m(a+x)\dot{\theta}^2$$

$$\frac{\partial V}{\partial \theta} = -mg(a+x)\cos\theta$$

$$\frac{\partial V}{\partial x} = kx - mg\sin\theta$$

Consequently, the equations of motion for this system are:

$$m(a+x)^2\ddot{\theta} + 2m(a+x)\dot{x}\dot{\theta} - mg(a+x)\cos\theta = 0$$
$$m\ddot{x} - m(a+x)\dot{\theta}^2 + kx - mg\sin\theta = 0$$



Dynamics 4
MECE10002
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Dynamics 4

Exercise Sheet 3

Notes for this Exercise Sheet The topics covered in this exercise sheet are the following: (i) analysis of continuous systems; dynamic stability (and instability). All these topics were discussed in the lectures and are covered in the lecture notes. If you want to read more about this or go through more examples you can, for example, see Sections 8.1 to 8.5 (pages 745 to 788) on the reference suggested for further reading[a]. Additionally, you can solve example 6.18 (page 656) and problems 8.22 to 8.47 on pages 809 to 811 of this same reference.

[a] S.S. Rao, Mechanical Vibrations (6^{th} edition, SI), Pearson, 2018 (ISBN 978-1-292-17860-8).

Question 3.1

A disc of negligible mass is spinning horizontally at ω rad/s. A mass M can slide frictionless along a slot that runs along the diameter of the disc as shown in Figure 1. The mass is held by two equal length and stiffness springs.

a) Derive the kinetic and potential energy functions for the system.
b) Use Lagrange's method to obtain an equation of motion for the radial motion of the mass.
c) Find a steady-state solution for this equation and determine its stability.
d) How does the mass behaviour change as the rotation speed is slowly increased?

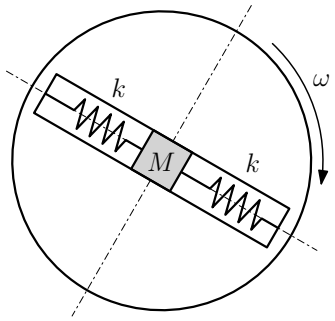


Figure 1: Rotating disc for Question 3.1.

Example solution

The system only has one degree-of-freedom since the mass is constrained to move inside the slot. This coordinate, r, is the radial distance of the mass M from the centre of the disc. Negative values of r mean that the displacement is in the opposite direction relative to the co-rotating frame.
The position of the mass M is given by

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

where $\theta = \omega t$ is the angle of rotation of the disc and $\dot{\theta} = \omega$ is the angular velocity. The linear components of the velocity are then

$$\dot{x} = \dot{r}\cos\theta - r\omega\sin\theta \quad \text{and} \quad \dot{y} = \dot{r}\sin\theta + r\omega\cos\theta$$

The kinetic energy of the system is

$$T = \frac{1}{2}M\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2}M\left(\dot{r}^2 + r^2\omega^2\right)$$

and the elastic potential energy of the two springs is

$$V = kr^2$$

Using Lagrange's equations, we obtain

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{r}}\right) - \frac{\partial T}{\partial r} + \frac{\partial V}{\partial r} = 0$$

$$M\ddot{r} - Mrw^2 + 2kr = 0$$

Reorganising terms, the equation of motion of the system becomes

$$M\ddot{r} + \left(2k - M\omega^2\right) r = 0$$

The steady-state solution can be obtained by setting the time derivatives to zero, leading to the trivial solution $r = 0$. The stability of the system is determined by the roots of the characteristic equation

$$M\lambda^2 + \left(2k - M\omega^2\right) = 0$$

which are

$$\lambda = \pm\sqrt{\frac{M\omega^2 - 2k}{M}} = \pm\sqrt{\omega^2 - \frac{2k}{M}}$$

For small values of ω, such that

$$\omega^2 < \frac{2k}{M}$$

the root is purely imaginary and the solution is stable (borderline stable). For rotation frequencies above the natural frequency of the non-rotating system, that is when

$$\omega^2 > \frac{2k}{M}$$

the solution becomes unstable and the mass moves radially outwards.

Question 3.2

The upright pendulum system shown in figure 2a can be used to model the dynamics of the vertical orientation of the behaviour of a spinning plate on a pole, such as the one shown in figure 2b. The pole has length l and mass m, and the plate is a lumped mass M fixed at the top of the pole.

a) Write the equation of motion of the upright pendulum system in figure 2a neglecting the boundary motion $\bar{z}(t)$ and all dissipative terms, such as friction.
b) Determine if the pendulum is stable or unstable in the upright position shown in figure 2.
c) Discuss the effect that increasing the length of the pole l or increasing the mass of the plate M will have on the dynamic stability of the system.
d) Write the equation of motion of the system considering the ground motion $\bar{z}(t)$ and discuss what effect this has on the stability of the pendulum in the upright position.

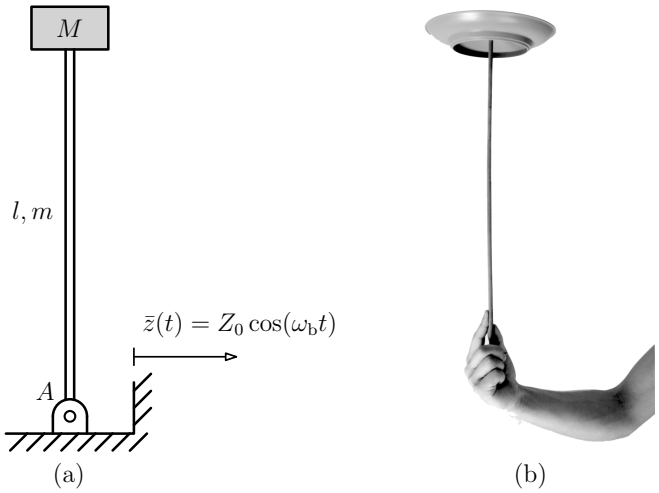


Figure 2: Spinning a plate on a pole for Question 3.2: (a) dynamic model and (b) spinning plate [Photo licence CC BY-SA 3.0].

Example solution

The generalised displacement of the system is the angle of rotation of the pendulum $\theta(t)$, measured from the upright position. Neglecting the boundary motion $\bar{z}(t)$ and noting that $I_M = Ml^2$ and $I_m = ml^2/3$, the equations for the kinetic energy T and potential energy V of the upright pendulum can be written as

$$T = \frac{1}{2} I_M \dot{\theta}^2 + \frac{1}{2} I_m \dot{\theta}^2 = \frac{1}{2} \left(M + \frac{m}{3} \right) l^2 \dot{\theta}^2$$

and

$$V = \left(M + \frac{m}{2} \right) gl \cos\theta$$

Using Lagrange's method, the equation of motion becomes

$$\left(M + \frac{m}{3} \right) l^2 \ddot{\theta} - \left(M + \frac{m}{2} \right) gl \sin\theta = 0$$

It should be noted that, in order to analyse the stability of the system, the equation of motion should not be linearised at this stage.
The upright stationary position is $\theta^s = 0$. The position $\theta^s = \pi$, although in theory also stationary, is irrelevant for the situation described. Consequently, when adding a perturbation to the angle, the angular velocity and acceleration become $\dot{\theta} = \delta\dot{\theta}$ and $\ddot{\theta} = \delta\ddot{\theta}$, respectively. Substituting in the equation of motion we obtain

$$\left(M + \frac{m}{3} \right) l^2 \delta\ddot{\theta} - \left(M + \frac{m}{2} \right) gl \sin(\delta\theta) = 0$$

Linearising the equation of motion around $\theta = 0$ leads to

$$\left(M + \frac{m}{3} \right) l^2 \delta\ddot{\theta} - \left(M + \frac{m}{2} \right) gl\delta\theta = 0$$



which has the following characteristic equation

$$\left(M + \frac{m}{3}\right) l^2\lambda^2 - \left(M + \frac{m}{2}\right) gl = 0$$

Solving for λ, the roots of the characteristic equation are

$$\lambda_{1,2} = \pm\sqrt{\frac{3(2M+m)}{2(3M+m)}\frac{g}{l}}$$

These roots are both real ($\mathrm{Re} \neq 0$) and one is positive ($\mathrm{Re} > 0$), meaning that the system is unstable in the upright position.
The stability of the system is defined by the two roots $\lambda_{1,2}$ (see above). As both these roots are real, increasing the length of the pole l will decrease the magnitude of the roots, bringing them closer to zero and, consequently, bringing the system closer to stability. In the theoretical limit case of an infinite length l, the system would become borderline stable ($\mathrm{Re} = 0$).
With the increase of mass of the plate, M, the ratio of the masses $(2M + m)/(3M + m)$ will decrease asymptotically to $2/3$, also leading to an increase in stability, although not as evident as with the increase of the length of the pole l — the system remains unstable.
Adding a horizontal boundary motion $\bar{z}(t)$ leads to a change in the kinetic energy of the system, which becomes

$$T = \frac{1}{2}M\left(\dot{x}_M^2 + \dot{y}_M^2\right) + \frac{1}{2}I_M\dot{\theta}^2 + \frac{1}{2}m\left(\dot{x}_m^2 + \dot{y}_m^2\right) + \frac{1}{2}I_m\dot{\theta}^2$$

where $I_m = ml^2/12$ is the moment of inertia of the pole about its centre of mass and $I_M = 0$ is the moment of inertia of the lumped mass M about its centre of mass. Also,

$$x_M = \bar{z} + l\sin\theta \quad \text{and} \quad y_M = l\cos\theta$$

$$x_m = \bar{z} + \frac{l}{2}\sin\theta \quad \text{and} \quad y_m = \frac{l}{2}\cos\theta$$

Note that the boundary motion $\bar{z}(t)$ only affects the horizontal components of the motion. Differentiating and substituting in T leads to

$$T = \frac{1}{2}M\left(\dot{\bar{z}}^2 + l^2\dot{\theta}^2 + 2\dot{\bar{z}}l\dot{\theta}\cos\theta\right) + \frac{1}{2}m\left(\dot{\bar{z}}^2 + \frac{1}{3}l^2\dot{\theta}^2 + \dot{\bar{z}}l\dot{\theta}\cos\theta\right)$$

The potential energy of the system does not change when considering the boundary motion $\bar{z}(t)$, thus

$$V = Mgl\cos\theta + mg\frac{l}{2}\cos\theta = \left(M + \frac{m}{2}\right) gl\cos\theta$$

Applying Lagrange's method we obtain the following equation of motion for the upright pendulum with a horizontal boundary motion

$$\left(M + \frac{1}{3}m\right) l^2\ddot{\theta} + \left(M + \frac{m}{2}\right) l\left(\ddot{\bar{z}}\cos\theta - g\sin\theta\right) = 0$$

which has the same characteristic equation as the system without boundary motion. Consequently, adding the boundary motion does not change the stability or instability characteristics of the system.

Question 3.3

A rigid bar AB is pivoted at point A and is mounted at the centre of a horizontal rotating disc as shown in figure 3.

a) Obtain the equation of motion for this system.
b) Find the corresponding steady-state solutions and analyse the stability of the system for different rotation velocities of the disc.

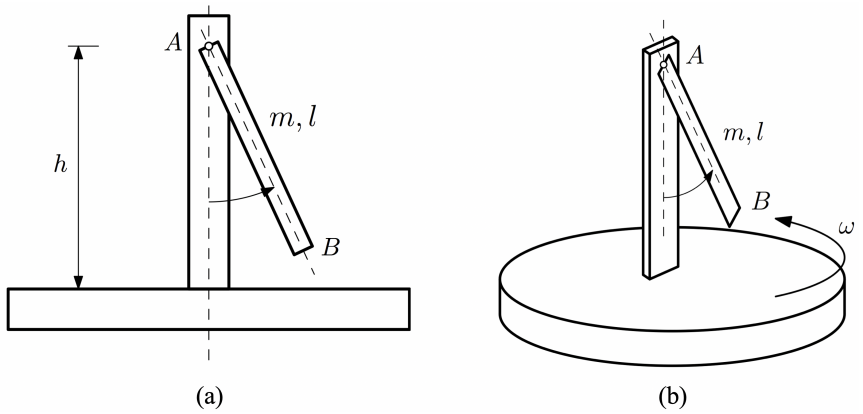


Figure 3: Rotating disc and oscillating bar system: (a) side and (b) 3D view.

Example solution

The system has one angular degree-of-freedom, $\theta(t)$. If h is the distance from point A to the disc, the coordinates of the centre of gravity of the oscillating bar are:

$$x_{\mathrm{G}} = \frac{l}{2}\sin\theta\cos\alpha, \qquad y_{\mathrm{G}} = \frac{l}{2}\sin\theta\sin\alpha, \qquad z_{\mathrm{G}} = h - \frac{l}{2}\cos\theta$$

The total kinetic energy of the system is

$$T = \frac{1}{2}m\left(\dot{x}_{\mathrm{G}}^2 + \dot{y}_{\mathrm{G}}^2 + \dot{z}_{\mathrm{G}}^2\right) + \frac{1}{2}I_{\mathrm{G}}\dot{\theta}^2 = \frac{1}{8}ml^2\left(\dot{\theta}^2 + \omega^2\sin^2\theta\right) + \frac{1}{24}ml^2\dot{\theta}^2 = \frac{1}{8}ml^2\left(\frac{4}{3}\dot{\theta}^2 + \omega^2\sin^2\theta\right)$$

In this solution we're neglecting the remaining two components of angular motion, as the corresponding moments of inertia will be significantly smaller than I_{G}. The potential energy is

$$V = mgz_{\mathrm{G}} = mg\left(h - \frac{l}{2}\cos\theta\right)$$

Using Lagrange's method, the equation of motion of the system can be written as

$$\frac{2}{3}ml^2\ddot{\theta} - \frac{1}{2}ml^2\omega^2\cos\theta\sin\theta + mgl\sin\theta = 0$$

The process of analysing the dynamic stability of a system and looking for its stationary positions can be done by forcing the kinematic terms in the equation of motion to be zero (i.e. *enforcing* the definition of "stationary"), leading to the following steady-state solutions — the stationary positions — for the system:

$$\theta = 0, \quad \theta = \pi \quad \text{and} \quad \cos\theta = \frac{2g}{\omega^2 l}$$

Note that for small angular velocities ω, the third condition cannot be fulfilled since $\cos\theta \leq 1$. As such, we are left with the first two stationary solutions

$$\theta^s = 0 \quad \text{and} \quad \theta^s = \pi$$

Applying a small perturbation to the first one ($\theta = 0 + \delta\theta$) leads to the following equation of motion:

$$\frac{2l}{3}\delta\ddot{\theta} + \left(g - \frac{l}{2}\omega^2\right)\delta\theta = 0$$

The roots of the corresponding characteristic equation are

$$\lambda_0 = \pm\sqrt{\frac{3}{2l}\left(\frac{l\omega^2 - 2g}{2}\right)} = \pm\sqrt{\frac{3}{4}\left(\omega^2 - \frac{2g}{l}\right)}$$

Consequently, the steady-state solution $\theta^s = 0$ may be stable or unstable, depending on the relation between $l\omega^2$ and $2g$. If $\omega^2 > 2g/l$ the roots λ_0 are real and the system is unstable. If, however, $\omega^2 < 2g/l$ the roots λ_0 are imaginary and the system becomes stable.
Applying a small perturbation to the second steady-state solution ($\theta = \pi + \delta\theta$) leads to the following equation of motion:

$$\frac{2l}{3}\delta\ddot{\theta} - \left(g + \frac{l}{2}\omega^2\right)\delta\theta = 0$$

Note that when linearising the equation of motion: $\sin(\pi + \delta\theta) \approx -\delta\theta$ and $\cos(\pi + \delta\theta) \approx -1$.
The roots of the corresponding characteristic equation are

$$\lambda_\pi = \pm\sqrt{\frac{3}{2l}\left(\frac{l\omega^2 + 2g}{2}\right)} = \pm\sqrt{\frac{3}{4}\left(\omega^2 + \frac{2g}{l}\right)}$$

The stability of the system is only assured if the roots of the characteristic equation are purely imaginary, that is, have no real part ($\mathrm{Re} = 0$). Consequently, as ω^2, g and l are always positive, the roots λ_π are always real. Thus, the steady-state solution $\theta^s = \pi$ is always unstable, as should be expected.

Question 3.4

The homogeneous L-bracket AOB shown in figure 4 has mass $2m$, is pivoted at point O and is connected to a fixed wall through a spring of stiffness k and a damper of damping constant c.

a) Obtain the equation of motion for this system.
b) Analyse the stability of the system if the spring is removed.

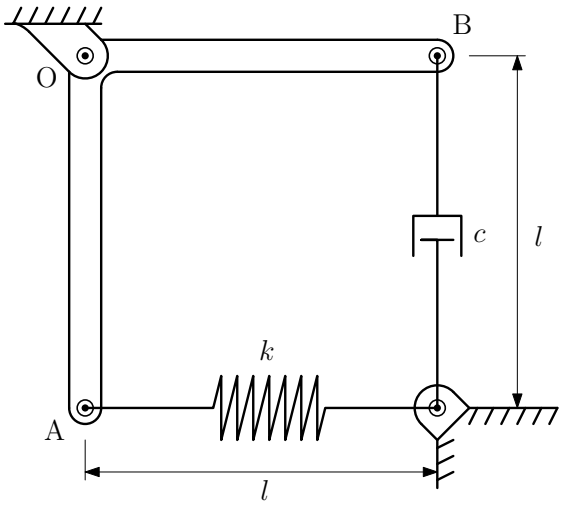


Figure 4: Oscillating bracket.

Example solution

The system has one angular degree-of-freedom, $\theta(t)$, measured around point O. The energy terms of the system can be calculated relative to the centre of mass of the whole bracket. An alternative solution, and probably a simpler/quicker one, is however to consider the L-bracket to be two distributed mass pendulums moving under the exact same generalised coordinate $\theta(t)$.
Following this approach, the total kinetic energy of the system is

$$T = 2 \times \frac{1}{2} I_{\mathrm{O}} \dot{\theta}^2 = I_{\mathrm{O}} \dot{\theta}^2 = \left(\frac{1}{3} m l^2 \right) \dot{\theta}^2$$

The potential energy is the sum of its elastic and gravitational components, $V = V_{\mathrm{e}} + V_{\mathrm{g}}$, where

$$V_{\mathrm{e}} = \frac{1}{2} k (l \sin\theta)^2 = \frac{1}{2} k l^2 \sin^2\theta$$

$$V_{\mathrm{g}} = \frac{1}{2} l m g \sin\theta - \frac{1}{2} l m g \cos\theta = \frac{1}{2} m g l \left(\sin\theta - \cos\theta\right)$$

The generalised force acting on the system, due to the damper, is

$$Q = -c l^2 \dot{\theta} \cos\theta$$

Applying Lagrange's method, the equation of motion of the system becomes

$$\frac{2}{3} m l^2 \ddot{\theta} + \frac{1}{2} m g l \left(\cos\theta + \sin\theta\right) + k l^2 \sin\theta \cos\theta = -c l^2 \dot{\theta} \cos\theta$$

As discussed in previous examples, we can analyse the dynamic stability of a system and find its stationary positions by forcing the kinematic terms in the equation of motion to be zero. In this example, if the spring is removed, the equation of motion becomes:

$$\frac{2}{3} m l^2 \ddot{\theta} + \frac{1}{2} m g l \left(\cos\theta + \sin\theta\right) = -c l^2 \dot{\theta} \cos\theta$$

The steady-state solutions of this equation can be determined as:

$$\frac{1}{2} m g l \left(\cos\theta + \sin\theta\right) = 0$$

which can be simplified to

$$\cos\theta = -\sin\theta \quad \Rightarrow \quad \tan\theta = -1$$

This stationary positions corresponds to values of the generalised coordinate of $\theta^{\mathrm{s}} = -\pi/4$ and $\theta^{\mathrm{s}} = 3\pi/4$. The following steps describe the stability analysis for $\theta^{\mathrm{s}} = -\pi/4$. Applying a small perturbation to the stationary position leads to the following equation of motion:

$$\frac{2}{3}ml^2\delta\ddot{\theta} + \frac{1}{2}mgl\left[\cos\left(-\frac{\pi}{4}+\delta\theta\right) + \sin\left(-\frac{\pi}{4}+\delta\theta\right)\right] = -cl^2\delta\dot{\theta}\cos\left(-\frac{\pi}{4}+\delta\theta\right)$$

The linearised versions of the $\sin(\cdot)$ and $\cos(\cdot)$ functions are:

$$\sin\theta \approx \sin\theta_0 + \frac{\mathrm{d}}{\mathrm{d}\theta}(\sin\theta)\bigg|_{\theta_0}(\theta-\theta_0) \quad \text{and} \quad \cos\theta \approx \cos\theta_0 + \frac{\mathrm{d}}{\mathrm{d}\theta}(\cos\theta)\bigg|_{\theta_0}(\theta-\theta_0)$$

Consequently, the $\sin(\cdot)$ and $\cos(\cdot)$ terms in the equation of motion become

$$\sin\left(-\frac{\pi}{4}+\delta\theta\right) \approx -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\delta\theta \quad \text{and} \quad \cos\left(-\frac{\pi}{4}+\delta\theta\right) \approx \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\delta\theta$$

The linearised version of the equation of motion then becomes

$$\frac{2}{3}ml^2\delta\ddot{\theta} + \frac{\sqrt{2}}{2}mgl\delta\theta = -\frac{\sqrt{2}}{2}cl^2\delta\dot{\theta}$$

The associated characteristic equation is

$$\frac{2}{3}ml^2\lambda^2 + \frac{\sqrt{2}}{2}mgl = -\frac{\sqrt{2}}{2}cl^2\lambda \quad \Leftrightarrow \quad \lambda^2 + \frac{3\sqrt{2}c}{4m}\lambda + \frac{3\sqrt{2}g}{4l} = 0$$

and the roots of this characteristic equation are

$$\lambda_{1,2} = -\frac{3\sqrt{2}c}{8m} \pm \sqrt{\left(\frac{3\sqrt{2}c}{8m}\right)^2 - \frac{3\sqrt{2}g}{l}}$$

Consequently, the stability of the steady-state solution $\theta^s = -\pi/4$ depends on relation between the magnitude of the terms

$$A = \frac{3\sqrt{2}c}{8m} \quad \text{and} \quad B = \frac{3\sqrt{2}g}{l}$$

- If the argument of the square root is zero, i.e. $A^2 = B$, which is equivalent to

  $$c = 3.88m\sqrt{\frac{g}{l}}$$

  then the characteristic equation of the system has a real and negative solution ($\mathrm{Re} < 0\ (\neq 0)\ \wedge\ \mathrm{Im} = 0$), meaning it is stable and non-oscillating (i.e. it converges to the stationary position without oscillations).
- If $A^2 < B$, which is equivalent to

  $$c < 3.88m\sqrt{\frac{g}{l}}$$

  then the characteristic equation has a complex solution, with a real and negative part ($\mathrm{Re} < 0\ (\neq 0) \wedge \mathrm{Im} \neq 0$), meaning the system is stable and will converge to the stationary position with oscillations.
- The case where $A^2 > B$ leads to two separate conditions: (i) one where $B < 0$, equivalent to

  $$\frac{3\sqrt{2}g}{l} < 0$$

  which is physically impossible, and (ii) another one where $B > 0$, which is the same as the first solution above, i.e. where $\mathrm{Re} < 0\ (\neq 0)\ \wedge\ \mathrm{Im} = 0$ and the system is stable.

In conclusion, when the spring is removed, regardless of the mass m, size l or damping constant c, the system will always be stable, albeit converging to the stationary position differently, which should be expected given the configuration of the L-bracket system.

Question 3.5

Consider the dynamic system in Figure 5. The rigid distributed mass pendulum AB has length L and mass m, and is pinned at point A. Mass M is connected to fixed boundaries through an elastic spring with stiffness k and a viscous damper with damping constant c. Force $P(t) = P_0 \cos(\omega t)$ is always horizontal and acts at point B, as shown in Figure 5.

a) Derive the equations of motion of the system and linearise them, assuming small vibrations. Do not neglect the effects of gravity.
b) Assume mass M is now fixed, and show that the pendulum in the hanging down position is borderline stable.

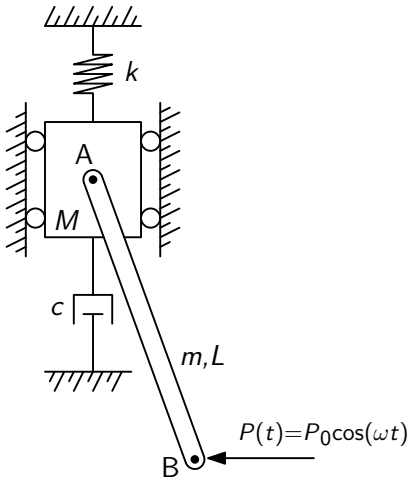


Figure 5: Dynamic system with oscillating mass and distributed mass pendulum.

Example solution

The system in Figure 5 has the following 2 generalised coordinates: (i) one to describe the vertical linear motion of mass M, which in this example solution will be labelled as $y(t)$, positive upwards, and (ii) the other to describe the angular motion of the distributed mass pendulum AB, labelled as $\theta(t)$, positive counter-clockwise.
The total kinetic energy of the system is

$$T = T_M + T_m = \frac{1}{2}M\dot{y}^2 + \frac{1}{2}m\left(\dot{x}_{\mathrm{G}}^2 + \dot{y}_{\mathrm{G}}^2\right) + \frac{1}{2}I_{\mathrm{G}}\dot{\theta}^2$$

where, G denotes the centre of mass of the pendulum and $I_{\mathrm{G}} = mL^2/12$ is the moment of inertia of the bar AB relative to its centre of mass. The coordinates of the centre of mass are

$$x_{\mathrm{G}} = \frac{1}{2}L\sin\theta \quad \text{and} \quad y_{\mathrm{G}} = y(t) - \frac{1}{2}L\cos\theta$$

Differentiating these coordinates to find the corresponding velocities yields

$$\dot{x}_{\mathrm{G}} = \frac{1}{2}L\dot{\theta}\cos\theta \quad \text{and} \quad \dot{y}_{\mathrm{G}} = \dot{y}(t) + \frac{1}{2}L\dot{\theta}\sin\theta$$

After substitution, the total kinetic energy becomes

$$T = \frac{1}{2}(M+m)\dot{y}^2 + \frac{1}{6}mL^2\dot{\theta}^2 + \frac{1}{2}mL\dot{y}\dot{\theta}\sin\theta$$

The total potential energy of the system is the sum of the gravitational and elastic components, that is, $V = V_{\mathrm{g}} + V_{\mathrm{e}}$. The gravitational potential energy is the sum of the gravitational energy for both the bar and the lumped mass, that is,

$$V_{\mathrm{g}} = Mgy + mg\left(y - \frac{1}{2}L\cos\theta\right)$$

The elastic potential energy of the system is

$$V_{\mathrm{e}} = \frac{1}{2}ky^2$$

The total potential energy is then

$$V = \frac{1}{2}ky^2 + Mgy + mg\left(y - \frac{1}{2}L\cos\theta\right)$$

to find the generalised forces, we consider that force $P(t) = P_0\cos(\omega t)$ is always horizontal and acts at point B. The component of this force that is perpendicular to the pendulum does work on the system and as such needs to be considered as a generalised force on the θ degree of freedom. Consequently, the generalised force (moment) on the pendulum is

$$Q_\theta = -LP_0\cos(\omega t)\cos\theta$$

Noting that the damper has one end fixed and acts directly on mass M, the corresponding generalised force is

$$Q_y = -c\dot{y}$$

As the external force $P(t)$ is always horizontal, it does not do work on mass M and as such does not contribute to the generalised force Q_y.
Using Lagrange's method, the equations of motion for the generalised coordinates $x(t)$ and $\theta(t)$ are, respectively,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{y}}\right) - \frac{\partial T}{\partial y} + \frac{\partial V}{\partial y} = Q_y \qquad \text{and}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} = Q_\theta$$

Differentiating the kinetic energy T and the potential energy V, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{y}}\right) = (M+m)\ddot{y} + \frac{1}{2}mL\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right), \qquad \frac{\partial T}{\partial y} = 0$$

$$\frac{\partial V}{\partial y} = ky + (M+m)g$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = \frac{1}{3}mL^2\ddot{\theta} + \frac{1}{2}mL\left(\ddot{y}\sin\theta + \dot{y}\dot{\theta}\cos\theta\right)$$

$$\frac{\partial T}{\partial \theta} = \frac{1}{2}mL\dot{y}\dot{\theta}\cos\theta, \qquad \frac{\partial V}{\partial \theta} = \frac{1}{2}mgL\sin\theta$$

After some arithmetic manipulation, the equations of motion of the system become

$$(M+m)\ddot{y} + \frac{1}{2}mL\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right) + ky + (M+m)g = -c\dot{y}$$

$$\frac{1}{3}mL^2\ddot{\theta} + \frac{1}{2}mL\ddot{y}\sin\theta + \frac{1}{2}mgL\sin\theta = -LP_0\cos(\omega t)\cos\theta$$

Linearising these equations of motion leads to

$$(M+m)\ddot{y} + \frac{1}{2}mL\ddot{\theta}\theta + ky + (M+m)g = -c\dot{y}$$

$$\frac{1}{3}mL^2\ddot{\theta} + \frac{1}{2}mL\ddot{y}\theta + \frac{1}{2}mgL\theta = -LP_0\cos(\omega t)$$

Note that the cross terms (on y and θ) should not be neglected as this would uncouple the equations of motion, although it could be argued that they are one order of magnitude smaller than all other terms.
If mass M is fixed it acts as a fixed boundary, i.e. $y(t) = 0$. For the hanging down position the pendulum has stationary coordinate $\theta_\mathrm{s} = 0$. Adding a perturbation to this stationary position yields

$$\theta = 0 + \delta\theta, \quad \dot{\theta} = \delta\dot{\theta} \quad \text{and} \quad \ddot{\theta} = \delta\ddot{\theta}$$

It should also be noted that, as mass M is fixed, $\ddot{y}(t) = 0$. Substituting in the non-linearised equation of motion for the generalised coordinate θ

$$\frac{1}{3}mL^2\delta\ddot{\theta} + \frac{1}{2}mgL\sin(\delta\theta) = -LP_0\cos(\omega t)\cos(\delta\theta)$$



Linearising the equation of motion, considering $\sin(\delta\theta) \approx \delta\theta$ and $\cos(\delta\theta) \approx 1$, leads to

$$\frac{1}{3}mL^2\delta\ddot{\theta} + \frac{1}{2}mgL\delta\theta = -LP_0\cos(\omega t)$$

The corresponding characteristic equation is

$$\frac{1}{3}mL^2\lambda^2 + \frac{1}{2}mgL = 0$$

Solving for the characteristic roots

$$\lambda^2 = \left(-\frac{1}{2}mgL\right) \Big/ \left(\frac{1}{3}mL^2\right) = -\frac{3g}{2L}$$

$$\lambda = \sqrt{-\frac{3g}{2L}} = \pm i\sqrt{\frac{3g}{2L}}$$

Both these roots are purely imaginary, i.e. with $\mathrm{Re} = 0$, meaning the system is borderline stable.

Dynamics 4

Exercise Sheet 4

Notes for this Exercise Sheet This exercise sheet covers: (i) analysis of continuous systems; (ii) shape functions and energy functions for continuous systems; (iii) Lagrange's method to obtain the equations of motion of continuous systems; (iv) generalised forces for continuous systems. All these topics were discussed in the lectures and are covered in the lecture notes. If you want to read more about this or go through more examples you can, for example, see Sections 8.1 to 8.5 (pages 745 to 788) on the reference suggested for further reading[a].

[a] S.S. Rao, Mechanical Vibrations (6th edition, SI), Pearson, 2018 (ISBN 978-1-292-17860-8).

Dynamics 4
MECE10002
Semester 1, 2026-27

Classroom
Classroom 12, Alrick

Learning module
LM4

Course notes
Section 2.5

Slides
LM4.3 to LM4.28

Discussion thread
continuous_systems

Question 4.1

Figure 1 shows an uniform cantilever beam. Obtain the approximate expression for the fundamental frequency of the beam using the static deflection polynomial

$$\varphi_{\mathrm{s}}(x) = \frac{2Lx^2 - x^3}{2L^3}$$

as the assumed mode shape function. Check if this complies with all relevant boundary conditions and compare with the result obtained for a quarter-cosine function

$$\varphi_{\mathrm{d}}(x) = 1 - \cos\left(\frac{\pi x}{2L}\right)$$

Which one is the best mode shape function? Why?

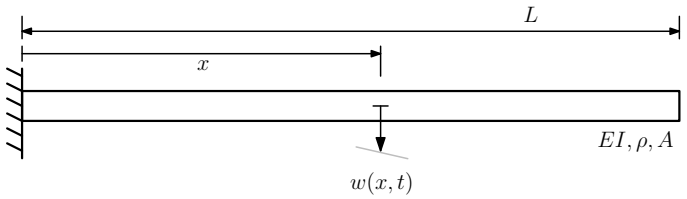


Figure 1: Cantilever beam for Question 4.1.

Example solution

To determine the fundamental frequency neglect gravity effects and focus only on the kinetic and elastic energy of the beam. These are given by

$$T = \frac{\rho A}{2}\dot{q}^2 \int_0^L [\varphi(x)]^2 \mathrm{d}x \quad \text{and} \quad V = \frac{EI}{2} q^2 \int_0^L \left[\frac{\partial^2 \varphi(x)}{\partial x^2}\right]^2 \mathrm{d}x$$

For the polynomial mode shape function — the static deflection polynomial — the kinetic energy becomes

$$T_1 = \frac{\rho A}{2}\dot{q}^2 \frac{1}{4L^6} \int_0^L \left(2Lx^2 - x^3\right)^2 \mathrm{d}x = \frac{1}{2}\left(\frac{29}{420}m\right)\dot{q}^2 = \frac{1}{2}\bar{M}_1 \dot{q}^2$$

and the potential energy becomes

$$V_1 = \frac{EI}{2} q^2 \frac{1}{4L^6} \int_0^L (4L - 6x)^2 \mathrm{d}x = \frac{1}{2}\left(\frac{EI}{L^3}\right) q^2 = \frac{1}{2}\bar{K}_1 q^2$$

For the trigonometric mode shape function — the quarter-cosine function — the kinetic energy expression becomes

$$T_2 = \frac{\rho A}{2}\dot{q}^2 \int_0^L \left[1 - \cos\left(\frac{\pi x}{2L}\right)\right]^2 \mathrm{d}x = \frac{\rho A}{2}\dot{q}^2 \left[\frac{3}{2} + \frac{1}{2}\cos\left(\frac{\pi x}{L}\right) - 2\cos\left(\frac{\pi x}{2L}\right)\right]_0^L$$

$$T_2 = \frac{1}{2}(0.227m)\dot{q}^2 = \frac{1}{2}\bar{M}_2 \dot{q}^2$$



and the elastic potential energy becomes

$$V_2 = \frac{EI}{2} q^2 \frac{\pi^4}{16L^4} \int_0^L \left[\cos^2\left(\frac{\pi x}{2L}\right)\right] \mathrm{d}x = \frac{1}{2}\left(\frac{\pi^4}{32}\frac{EI}{L^3}\right) q^2 = \frac{1}{2}\bar{K}_2 q^2$$

consequently, the natural frequency obtained with the polynomial mode shape function is

$$\omega_0^2 = \frac{\bar{K}_1}{\bar{M}_1} = \frac{420}{29m}\frac{EI}{L^3} = 14.5\frac{EI}{mL^3}$$

With the trigonometric mode shape function is

$$\omega_0^2 = \frac{\bar{K}_2}{\bar{M}_2} = \frac{\pi^4}{32 \times 0.227m}\frac{EI}{L^3} = 13.4\frac{EI}{mL^3}$$

The trigonometric shape function produces the better result. According to Rayleigh's theorem this method will always produce results above (or equal to) the true value. Consequently, the lower result will be the better one. It should also be noted that the static deflection polynomial $\varphi_\mathrm{s}(x)$ does not comply with all relevant boundary conditions: the bending moment at $x = L$ is not zero.

Question 4.2

Figures 2(a) and (b) show two cases of distributed pressure load that is suddenly applied to the top surface of a uniform cantilever beam. Using the same mode shape functions as in the previous question, estimate the ratio of the maximum tip deflections produced by the two cases. Think about the generalised forces acting in both cases and try to keep calculations to a minimum.

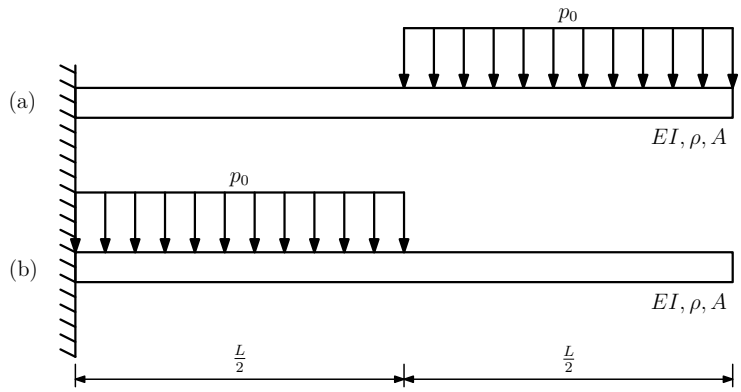


Figure 2: Cantilever beam with applied distributed pressure loads.

Example solution

The deflections w_a and w_b will be proportional to the applied loads Q_a and Q_b, respectively. Based on the superposition principle[1] this relation between w and Q will be the same for these two cases, which can be expressed as:

$$\frac{w_a}{w_b} = \frac{Q_a}{Q_b}$$

This principle states that on a linear elastic structure the effect of several simultaneous loads equals the effect of the sum of each of the loads. Therefore, to find the ratio of the maximum tip deflections we just need to calculate the ratio of the generalised forces acting on the beam. Using mode shape function

$$\varphi_1 = \frac{2Lx^2 - x^3}{2L^3}$$

for case (a) we get

$$\begin{aligned} Q_a &= \int_{L/2}^{L} p_0 \varphi_1(x) dx = p_0 \left[\frac{1}{2L^3} \left(\frac{2L}{3} x^3 - \frac{1}{4} x^4 \right) \right]_{L/2}^{L} \\ &= p_0 \left[\frac{1}{2L^3} \left(\frac{2L}{3} L^3 - \frac{1}{4} L^4 \right) \right] - p_0 \left[\frac{1}{2L^3} \left(\frac{2L}{3} \left(\frac{L}{2} \right)^3 - \frac{1}{4} \left(\frac{L}{2} \right)^4 \right) \right] \\ &= p_0 \left[\left(\frac{L}{3} - \frac{L}{8} \right) \right] - p_0 \left[\frac{L}{24} - \frac{L}{128} \right] \\ &= p_0 L \left[\frac{5}{24} - \frac{13}{384} \right] \approx 0.174 p_0 L \end{aligned}$$

and for case (b)

$$\begin{aligned} Q_b &= \int_{0}^{L/2} p_0 \varphi_1(x) dx = p_0 \left[\frac{1}{2L^3} \left(\frac{2L}{3} x^3 - \frac{1}{4} x^4 \right) \right]_{0}^{L/2} \\ &= p_0 \left[\frac{1}{2L^3} \left(\frac{2L}{3} \left(\frac{L}{2} \right)^3 - \frac{1}{4} \left(\frac{L}{2} \right)^4 \right) \right] \\ &= p_0 L \left[\frac{1}{24} - \frac{1}{128} \right] = p_0 L \left(\frac{13}{384} \right) \approx 0.034 p_0 L \end{aligned}$$

[1] Think back to Structural Mechanics 2 and see `https://edin.ac/4gSwWIC`

The ratio of the maximum tip deflection is hence

$$\frac{w_a}{w_b} = \frac{Q_a}{Q_b} \approx \frac{0.174}{0.034} \approx 5.12$$

Using mode shape function

$$\varphi_2 = 1 - \cos\left(\frac{\pi x}{2L}\right)$$

for case (a) we obtain

$$\begin{aligned} Q_a &= \int_{L/2}^{L} p_0 \varphi_2(x) dx = p_0 \left[\left(x - \frac{2L}{\pi}\sin(\frac{\pi x}{2L})\right)\right]_{L/2}^{L} \\ &= p_0 \left[\left(L - \frac{2L}{\pi}\sin\left(\frac{\pi}{2}\right)\right)\right] - p_0 \left[\left(\frac{L}{2} - \frac{2L}{\pi}\sin\left(\frac{\pi}{4}\right)\right)\right] \\ &= p_0 L \left[\left(1 - \frac{2}{\pi}\right)\right] - p_0 L \left[\left(\frac{1}{2} - \frac{2}{\pi}0.71\right)\right] \\ &= p_0 L \left[0.363 - 0.048\right] = 0.315 p_0 L \end{aligned}$$

and for case (b)

$$\begin{aligned} Q_b &= \int_{0}^{L/2} p_0 \varphi_2(x) dx = p p_0 \left[\left(x - \frac{2L}{\pi}\sin(\frac{\pi x}{2L})\right)\right]_{0}^{L/2} \\ &= p_0 \left[\left(\frac{L}{2} - \frac{2L}{\pi}\sin\left(\frac{\pi}{4}\right)\right)\right] \\ &\approx p_0 L \left[\left(\frac{1}{2} - \frac{2}{\pi}0.71\right)\right] \\ &\approx 0.048 p_0 L \end{aligned}$$

The ratio of the maximum tip deflection is then

$$\frac{w_a}{w_b} = \frac{Q_a}{Q_b} \approx \frac{0.315}{0.048} \approx 6.563$$



Question 4.3

A uniform beam is supported by end springs and has a damper attached at its midpoint as shown in Figure 3. Obtain the generalised force terms that must be added to the beam equations using the following mode shape functions:

$$\varphi_1(x) = 1$$
$$\varphi_2(x) = \cos\left(\frac{\pi x}{L}\right)$$

Write down the equations of motion of the system.

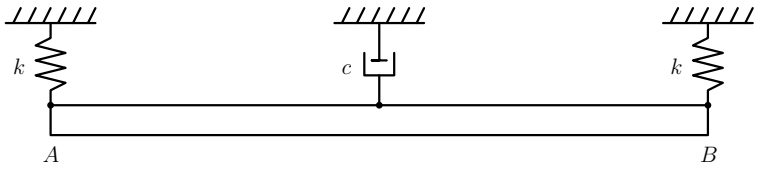


Figure 3: Beam supported on springs for Question 4.3.

Example solution

We start by determining the generalised forces on the system, due to the presence of the viscous damper. Choosing a coordinate system with its origin at the centre of the beam, such that the beam midpoint has coordinate $x = 0$. The mode shape functions are

$$\varphi_1(x) = 1 \quad \text{and} \quad \varphi_2(x) = \cos\left(\frac{\pi x}{L}\right)$$

and the deflection of the beam is

$$w(x,t) = q_1(t)\varphi_1(x) + q_2(t)\varphi_2(x)$$

where $q_1(t)$ and $q_2(t)$ are the generalised coordinates of the system. Recalling that the damper is positioned at the centre of the beam the damper force is given by

$$F_\mathrm{d} = -c\dot{w}(0,t)$$

Increasing the first generalised coordinate by δq_1 and keeping the second fixed produces a displacement of the beam midpoint of $\delta w(0) = \delta q_1 \varphi_1(0) = \delta q_1$. Accordingly, the damper force does a virtual work $\delta W = F_\mathrm{d}\delta q_1 = -c\left(\dot{q}_1 + \dot{q}_2\right)\delta q_1$. Hence, the generalised force on the first coordinate is

$$Q_1 = -c(\dot{q}_1 + \dot{q}_2)$$

Increasing the second generalised coordinate by δq_2 and keeping the first fixed produces a displacement of the beam midpoint of $\delta w(0) = \delta q_2 \varphi_2(0) = \delta q_2$ and a virtual work $\delta W = F_\mathrm{d}\delta q_2 = -c\left(\dot{q}_1 + \dot{q}_2\right)\delta q_2$. Hence the generalised force on the second coordinate is also

$$Q_2 = -c(\dot{q}_1 + \dot{q}_2) = Q_1$$

To determine the equations of motion we first calculate the total kinetic energy of the beam, with the suggested mode shape functions is

$$\begin{aligned} T &= \frac{1}{2}\rho A \int_{-L/2}^{L/2} \left[\dot{w}(x,t)\right]^2 \mathrm{d}x = \frac{1}{2}\rho A \int_{-L/2}^{L/2} \left[\dot{q}_1(t)\varphi_1(x) + \dot{q}_2(t)\varphi_2(x)\right]^2 \mathrm{d}x \\ &= \frac{1}{2}\rho A \left[\dot{q}_1^2 \int_{-L/2}^{L/2} \varphi_1^2 \mathrm{d}x + \dot{q}_2^2 \int_{-L/2}^{L/2} \varphi_2^2 \mathrm{d}x + 2\dot{q}_1\dot{q}_2 \int_{-L/2}^{L/2} \varphi_1\varphi_2 \mathrm{d}x\right] \\ &= \frac{1}{2}\rho A L \left(\dot{q}_1^2 + \frac{\dot{q}_2^2}{2} + \frac{4}{\pi}\dot{q}_1\dot{q}_2\right) = \frac{1}{4}m\left(2\dot{q}_1^2 + \dot{q}_2^2 + \frac{8}{\pi}\dot{q}_1\dot{q}_2\right) \end{aligned}$$

Assuming small vertical displacements we can neglect the gravitational energy terms. The total elastic potential energy is then

$$\begin{aligned} V &= \frac{1}{2}EI \int_{-L/2}^{L/2} \left[w''(x,t)\right]^2 \mathrm{d}x = \frac{1}{2}EI \int_{-L/2}^{L/2} \left[q_1(t)\varphi_1''(x) + q_2(t)\varphi_2''(x)\right]^2 \mathrm{d}x \\ &= \frac{1}{2}EI \left[q_1^2 \int_{-L/2}^{L/2} \left(\varphi_1''\right)^2 \mathrm{d}x + q_2^2 \int_{-L/2}^{L/2} \left(\varphi_2''\right)^2 \mathrm{d}x + 2q_1q_2 \int_{-L/2}^{L/2} \varphi_1''\varphi_2'' \mathrm{d}x\right] \\ &= \frac{\pi^4 EI}{4L^3} q_2^2 \end{aligned}$$

Note that in the equation for V above, we are not considering the contribution of the elastic energy of the springs as, in this case, the springs are the supports and not part of the structure. This is different from what we did when analysing discrete systems, where masses were rigid bodies and springs/dampers were considered to be components of the system — in this module, we are focusing on the elastic energy of deformable structures.
Finally, deriving the equations of motion using Lagrange's method leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_1}\right)+\frac{\partial V}{\partial q_1}=Q_1 \quad \Rightarrow \quad m\left(\ddot{q}_1+\frac{2}{\pi}\ddot{q}_2\right)=-c\left(\dot{q}_1+\dot{q}_2\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_2}\right)+\frac{\partial V}{\partial q_2}=Q_2 \quad \Rightarrow \quad m\left(\frac{2}{\pi}\ddot{q}_1+\frac{\ddot{q}_2}{2}\right)+\frac{\pi^4 EI}{2L^3}q_2=-c\left(\dot{q}_1+\dot{q}_2\right)$$

The equation of motion for the first mode, associated to generalised coordinate q_1, only has inertial terms because this is a rigid body motion mode shape. As expected, the equation of motion for q_2 has the stiffness term associated to the beam bending.

Question 4.4

The diving board shown in Figure 4 is modelled as a beam with a clamped end at $x = 0$ and a free end at $x = l$, where a diver with mass M stands. The mass of the board is $m = 30$ kg, the length is $l = 2$ m and the bending stiffness is $EI = 2 \times 10^4$ Nm^2.

a) Specify a shape function $\varphi(x)$ to describe the vibration behaviour of the lowest frequency mode of the system. Give reasons for your choice.
b) Using this shape function evaluate the corresponding kinetic and potential energy functions.
c) Use Lagrange's equations to obtain the equation of motion (for the lowest frequency mode).
d) Determine the mode frequency of the board alone and considering a diver standing on its free end ($M = 80$ kg).

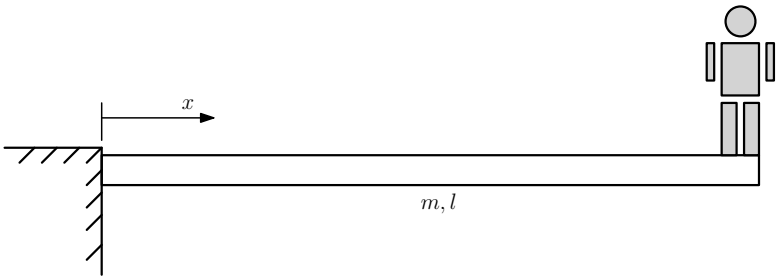


Figure 4: Diving board model for Question 4.4.

Example solution

A possible shape function for the lowest frequency vibration mode is

$$\varphi(x) = \left(\frac{x}{l}\right)^2$$

It should be noted that this shape function verifies the boundary conditions at $x = 0$ but it does not ensure the correct bending moment on the beam (for this we would need a cubic shape function). Using the above shape function, the energy expressions become

$$T = \frac{1}{2}\rho A\dot{q}^2 \int_0^l \varphi(x)^2 \mathrm{d}x + \frac{1}{2}M\dot{q}^2 = \frac{1}{2}\left(\frac{m}{5} + M\right)\dot{q}^2$$

$$V = Mgq + \rho Agq \int_0^l \varphi(x)\mathrm{d}x + \frac{1}{2}EIq^2 \int_0^l \left[\frac{\partial^2 \varphi(x)}{\partial x^2}\right]^2 \mathrm{d}x = Mgq + \frac{\rho Al}{3}gq + \frac{2EI}{l^3}q^2$$

The equation of motion of the system is

$$\left(\frac{m}{5} + M\right)\ddot{q} + \frac{4EI}{l^3}q + \left(\frac{m}{3} + M\right)g = 0$$

Substituting known values for l, EI and the masses m and M we get the following equation of motion

$$86\ddot{q} + 10^4 q + 883 = 0$$

which can be written in the generic form

$$\bar{M}\ddot{q} + \bar{K}q + 883 = 0$$

where $\bar{M}$ and $\bar{K}$ are the effective mass and stiffness of the system, that is, the mass and stiffness of an equivalent single degree of freedom system. The natural frequency of the board considering a person standing on its free end is then

$$\omega_n = \sqrt{\frac{\bar{K}}{\bar{M}}} = \sqrt{\frac{10^4}{86}} = 10.8 \text{ rad/s}$$

Setting $M = 0$ on the equation of motion we get the natural frequency of the board alone

$$\omega_n = \sqrt{\frac{\bar{K}}{\bar{M} - M}} = \sqrt{\frac{10^4}{6}} = 40.8 \text{ rad/s}$$

Dynamics 4

Exercise Sheet 5

Notes for this Exercise Sheet The topics covered in this exercise sheet are: (i) dynamics of continuous systems; (ii) wave equations; (iii) longitudinal waves on a uniform bar; (iv) travelling wave solutions; (v) standing-wave solutions. All these topics were discussed in the lectures and are covered in the lecture notes. If you want to read more about this or go through more examples you can, for example, see Sections 8.1 to 8.5 (pages 745 to 788) on the reference suggested for further reading[a]. Additionally, you can solve example 6.18 (page 656) and problems 8.22 to 8.47 on pages 809 to 811 of this same reference.

[a] S.S. Rao, Mechanical Vibrations (6th edition, SI), Pearson, 2018 (ISBN 978-1-292-17860-8).

Dynamics 4
MECE10002
Semester 1, 2026-27

Classroom
Classroom 12, Alrick

Learning module
LM5

Course notes
Section 3.1
Section 3.2

Slides
LM5.3 to LM5.14

Discussion thread
longitudinal_waves

Question 5.1

Follow the method of the bar example done in class to find expressions for the set of natural frequencies and wave lenghts of a wire of length L tightly stretched between two fixings. Wire has tension T and mass per unit length μ.

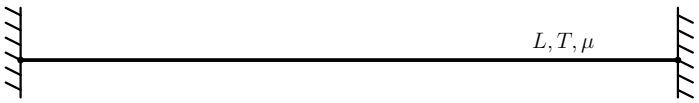


Figure 1: Stretched wire for Question 5.1.

Example solution

The boundary conditions for the stretched wire are $w(0,t) = w(L,t) = 0$[1] and the longitudinal wave equation is

$$\frac{\partial^2 w}{\partial t^2} = \frac{T}{\mu}\frac{\partial^2 w}{\partial x^2}$$

where T [N] is the tension of the wire and μ [kg/m] is its mass per unit length. Thus, $T/\mu = c^2$ [(m/s)^2]. Assuming a standing wave solution, that is, $w(x,t) = f(x)g(t)$ and inserting into the wave equation leads to

$$\frac{\partial^2 g}{\partial t^2} + \omega^2 g = 0$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\omega^2}{c^2} f = 0$$

The solution of the shape function $f(x)$ is

$$f(x) = A\cos\left(\frac{\omega x}{c}\right) + B\sin\left(\frac{\omega x}{c}\right)$$

Considering the boundary condition at $x = 0$ yields

$$w(0,t) = 0 \Rightarrow f(0) = 0 \Rightarrow A = 0$$

and at $x = L$ yields

$$w(L,t) = 0 \Rightarrow f(L) = 0 \Rightarrow B\sin\left(\frac{\omega L}{c}\right) = 0$$

From this boundary condition

$$\frac{\omega_n L}{c} = n\pi \quad \text{with} \quad n = 1, \ldots$$

[1] Note that in cables under tension, although the load is always aligned with the cable line (i.e. longitudinal), we describe the wave function as $w(x,y)$, instead of $u(x,t)$, as the resulting waves are of transverse displacement.



Thus, the natural frequencies of the wire become

$$\omega_n = n\frac{\pi}{L}\sqrt{\frac{T}{\mu}} \quad \text{with} \quad n = 1, \dots$$

and the wave lengths

$$\lambda_n = 2\pi\frac{c}{\omega_n} = \frac{2L}{n} \quad \text{with} \quad n = 1, \dots$$

Question 5.2

A steel bar has length $L = 3.5$ m and diameter $d = 45$ mm. This bar is suspended horizontally from two long wires fixed to the ceiling so that both ends of the bar are free to move. A small accelerometer (with negligible mass) is fixed at one end and the opposite end of the bar is tapped with a hammer. What frequencies would you expect to be registered by the accelerometer. List the first four in Hz and consider $E = 207$ GPa and $\rho = 7830$ kg/m^3.

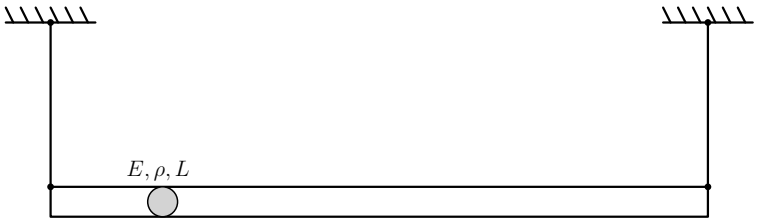


Figure 2: Suspended steel bar for Question 5.2.

Example solution

As the steel bar is suspended both ends are stress-free and the boundary conditions for the bar are $u'(0,t) = u'(L,t) = 0$. The longitudinal wave equation is

$$\frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho}\frac{\partial^2 u}{\partial x^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

Assuming a standing wave solution, that is, $u(x,t) = f(x)g(t)$ and inserting into the wave equation leads to

$$\frac{\partial^2 g}{\partial t^2} + \omega^2 g = 0$$
$$\frac{\partial^2 f}{\partial x^2} + \frac{\omega^2}{c^2} f = 0$$

The solution of the shape function $f(x)$ is

$$f(x) = A\cos\left(\frac{\omega x}{c}\right) + B\sin\left(\frac{\omega x}{c}\right)$$

The first derivative relative to x of the displacement function is $u'(x,t) = f'(x)g(t)$ and

$$f'(x) = -\frac{\omega}{c}A\sin\left(\frac{\omega x}{c}\right) + \frac{\omega}{c}B\cos\left(\frac{\omega x}{c}\right)$$

Defining two new constants as

$$C = -\frac{\omega}{c}A \quad \text{and} \quad D = \frac{\omega}{c}B$$

the derivative of the shape function can be written as

$$f'(x) = C\sin\left(\frac{\omega x}{c}\right) + D\cos\left(\frac{\omega x}{c}\right)$$

Considering the boundary condition for $x = 0$ yields

$$u'(0,t) = 0 \Rightarrow f'(0) = 0 \Rightarrow D = 0 \Rightarrow B = 0$$

and for $x = L$ yields

$$u'(L,t) = 0 \Rightarrow f'(L) = 0 \Rightarrow C\sin\left(\frac{\omega L}{c}\right) = 0$$

From this boundary condition

$$\frac{\omega_n L}{c} = n\pi \quad \text{with} \quad n = 1, \ldots$$

Thus, the natural frequencies of the steel bar become

$$\omega_n = n\frac{c\pi}{L} = n\frac{\pi}{L}\sqrt{\frac{E}{\rho}} \quad n = 1, \ldots$$



and the mode shapes (wave lengths)

$$\lambda_n = 2\pi\frac{c}{\omega_n} = \frac{2L}{n} \quad \text{with} \quad n = 1, \dots$$

The first four frequencies and wavelengths are

n	ω_n [rad/s]	ω_n [Hz]	λ_n [m]
1	4615	734.5	7.00
2	9230	1469.0	3.50
3	13845	2203.5	2.33
4	18461	2938.2	1.75

Question 5.3

A uniform bar of cross-section area A, length l and Young's modulus E is connected at both ends by springs, viscous dampers and masses as shown in Figure 3. State the appropriate boundary conditions for this system.

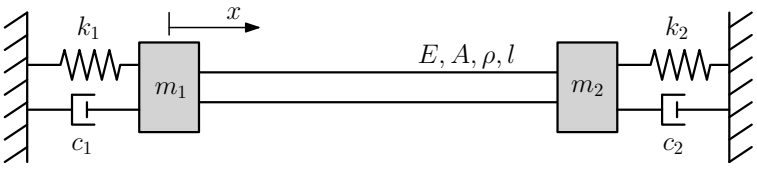


Figure 3: Boundary conditions for Question 5.3.

Example solution

The external forces acting on mass m_1 are due to the spring $(-k_1 u)$, the viscous damper $(-c_1 \dot{u})$ and the inertia of the lumped mass $(-m_1 \ddot{u})$. There is, however, an additional force due to the stress equilibrium at the interface between the mass and the bar. This internal force is

$$AE\frac{\partial u}{\partial x}$$

The external forces acting on mass m_2 are similar, that is, due to the spring $(-k_2 u)$, the viscous damper $(-c_2 \dot{u})$, the inertia of the mass $(-m_2 \ddot{u})$ and the internal force due to the stress equilibrium between the mass and the bar. As the bar needs to be in equilibrium, this internal force is an action-reaction pair to the one on the opposite end of the bar, which naturally has opposite sign,

$$-AE\frac{\partial u}{\partial x}$$

Writing the equilibrium of the internal and external forces at $x = 0$, we get the first boundary condition for the bar

$$AE\frac{\partial u}{\partial x}(0,t) = -k_1 u(0,t) - c_1\frac{\partial u}{\partial t}(0,t) - m_1\frac{\partial^2 u}{\partial t^2}(0,t)$$

Identically, by writing the equilibrium of the internal and external forces at $x = L$, we get the second boundary condition for the bar

$$AE\frac{\partial u}{\partial x}(L,t) = k_2 u(L,t) + c_2\frac{\partial u}{\partial t}(L,t) + m_2\frac{\partial^2 u}{\partial t^2}(L,t)$$

Question 5.4

Using a standing-wave approach, find the free-vibration solution and natural frequencies of a bar of length l fixed at one end and free at the other (see Figure 4).

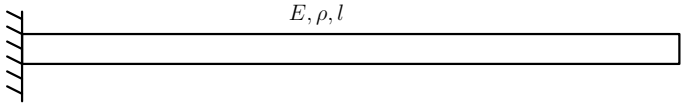


Figure 4: Fixed bar for Question 5.4.

Example solution

The bar is fixed at its left end ($x = 0$) and the corresponding boundary condition is $u(0,t) = 0$. The right end of the bar ($x = l$) is stress free and, consequently, the corresponding boundary condition is

$$\frac{\partial u(l,t)}{\partial x} = 0.$$

The longitudinal (axial) wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

Considering a standing wave solution for the wave equation

$$u(x,t) = X(x)T(t) = \left[C_1 \cos\left(\frac{\omega x}{c}\right) + C_2 \sin\left(\frac{\omega x}{c}\right)\right] \left[C_3 \cos(\omega t) + C_4 \sin(\omega t)\right]$$

and inserting the first boundary condition $u(0,t) = 0$ leads to $C_1 = 0$. Thus,

$$u(x,t) = C_2 \sin\left(\frac{\omega x}{c}\right) \left[C_3 \cos(\omega t) + C_4 \sin(\omega t)\right]$$

and

$$\frac{\partial u(x,t)}{\partial x} = \frac{C_2 \omega}{c} \cos\left(\frac{\omega x}{c}\right) \left[C_3 \cos(\omega t) + C_4 \sin(\omega t)\right].$$

Inserting the second boundary condition (at $x = l$) leads to

$$\frac{\partial u(l,t)}{\partial x} = 0 \quad \Rightarrow \quad \frac{\omega_n l}{c} = \frac{\pi}{2} + n\pi \quad \text{with} \quad n = 0, \ldots$$

and, rearranging terms, the natural frequencies of the bar become

$$\omega_n = \frac{(1+2n)}{2} \frac{\pi c}{l} \quad \text{with} \quad n = 0, \ldots$$

Question 5.5

Consider a bar of length l with one end fixed and a mass M attached at the other end, as in Figure 5. Find the first and second natural frequencies of the bar as a function of the ratio β between the mass of the bar (m) and the mass M (that is, $\beta = m/M$).

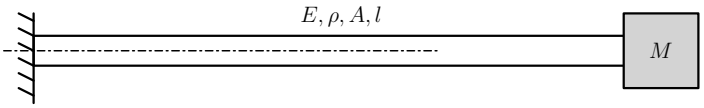


Figure 5: Fixed bar with attached mass for Question 5.5.

Example solution

The boundary condition for the bar at $x = 0$ (no displacement) is

$$u(0,t) = 0$$

and for $x = l$ (inertia due to the mass M) is

$$AE\frac{\partial u}{\partial x}(l,t) = -M\frac{\partial^2 u}{\partial t^2}(l,t).$$

The longitudinal wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}.$$

Assuming a standing wave solution $u(x,t) = X(x)T(t)$ for the wave equation, the solution is

$$u(x,t) = \sin\left(\frac{\omega x}{c}\right)\left[C\cos(\omega t) + D\sin(\omega t)\right].$$

Applying the boundary condition for $x = l$ leads to

$$AE\frac{\omega}{c}\cos\left(\frac{\omega l}{c}\right)\left[C\cos(\omega t) + D\sin(\omega t)\right] = M\omega^2\sin\left(\frac{\omega l}{c}\right)\left[C\cos(\omega t) + D\sin(\omega t)\right],$$

that simplifies to

$$\frac{AE}{c}\cos\left(\frac{\omega l}{c}\right) = M\omega\sin\left(\frac{\omega l}{c}\right)$$

By multiplying both sides of the above equation by $\rho L/M$ and recalling that $\rho AL = m$ and $c^2 = E/\rho$, we obtain,

$$\frac{\omega l}{c}\tan\left(\frac{\omega l}{c}\right) = \frac{m}{M} = \beta.$$

Considering $\alpha = \omega l/c$ the previous expression becomes $\tan(\alpha) = \frac{\beta}{\alpha}$. It is now possible to plot functions β/α and $\tan(\alpha)$ and determining the interceptions for different values of the mass ratio β. The table below lists some of these results, for the first two wave frequencies.

Mass ratio β	$\alpha_1 = \omega_1 l/c$	$\alpha_2 = \omega_2 l/c$
0.01	0.1	3.1448
0.1	0.3113	3.1736
1	0.8602	3.4267
10	1.4291	4.3063
100	1.5549	4.6658

Question 5.6

A bar of uniform cross-sectional area A, density ρ, Young's modulus E and length l is fixed at one end and free at the opposite end. This bar is subjected to an axial force F_0 at its free end, as shown in Figure 6. Study the vibrations of the bar if the force F_0 is suddenly removed.

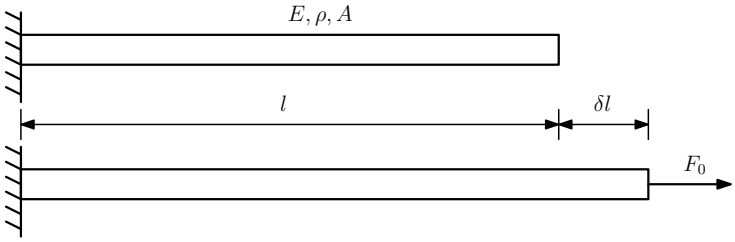


Figure 6: Fixed bar with axial force for Question 5.6.

Example solution

The bar is fixed at its left end ($x = 0$) and the corresponding boundary condition is $u(0, t) = 0$. The right end of the bar has an applied axial force F_0. The corresponding strain is

$$\varepsilon = \frac{F_0}{EA}.$$

Another boundary condition can be obtained from the axial displacement field before removing the force, that is

$$u_0 = u(x, 0) = \varepsilon x = \frac{F_0 x}{EA} \quad \text{for} \quad 0 \leq x \leq l.$$

An additional boundary condition derives from the fact that the bar is initially static (zero velocity), that is

$$\dot{u}_0 = \frac{\partial u(x, 0)}{\partial t} = 0 \quad \text{for} \quad 0 \leq x \leq l.$$

Using a standing wave solution expanded to include an infinite number of terms, the displacement field can be described by[2,3]

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t) = \sum_{n=0}^{\infty} f_n(x) g_n(t) =$$

$$= \sum_{n=0}^{\infty} \sin\left[\frac{(2n+1)\pi x}{2l}\right] \left\{ C_n \cos\left[\frac{(2n+1)\pi ct}{2l}\right] + D_n \sin\left[\frac{(2n+1)\pi ct}{2l}\right] \right\}.$$

Note that making $n = 0$ in the above equation makes this solution the same as the single mode shape approximation. Obtaining the time derivative of the displacement function and considering the velocity boundary condition $\dot{u}_0 = 0$ it is possible to say that $D_n = 0$. The displacement field becomes

$$u(x, t) = \sum_{n=0}^{\infty} C_n \sin\left[\frac{(2n+1)\pi x}{2l}\right] \cos\left[\frac{(2n+1)\pi ct}{2l}\right].$$

The constants C_n on the previous function are

$$C_n = \frac{2}{l} \int_0^l u_0 \sin\left[\frac{(2n+1)\pi x}{2l}\right] \mathrm{d}x = \frac{(-1)^n}{(2n+1)^2} \frac{8F_0 l}{EA\pi^2}.$$

Then

$$u(x, t) = \frac{8F_0 l}{EA\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \sin\left[\frac{(2n+1)\pi x}{2l}\right] \cos\left[\frac{(2n+1)\pi ct}{2l}\right].$$

The natural frequencies can be obtained from the cosine function

$$\omega_n = \frac{(2n+1)\pi c}{2l} \quad \text{with} \quad n = 0, \ldots$$

[2] Think back to Structural Mechanics and Dynamics 3.

[3] Although the level of detail in this solution is not covered in Dynamics 4, you can find the detailed derivation in S. Rao, *Mechanical Vibrations* (6th Edition), Prentice Hall, 2010 (Section 8.3).

Dynamics 4

Exercise Sheet 6

Notes for this Exercise Sheet The topics covered in this exercise sheet are: (i) dynamics of continuous systems; (ii) wave equations; (iii) transverse (flexural) waves on a uniform beam; (iv) travelling and standing-wave solutions. All these topics were discussed in the lectures and are covered in the lecture and class notes. If you want to read more about this or go through more examples you can, for example, see Section 8.5 (pages 767 to 788) on the reference suggested for further reading[a]. Additionally, you can go through examples 8.7 to 8.9 (pages 774 and 780) and solve problems 8.30 to 8.55 on pages 809 to 813 of this same reference.

[a] S.S. Rao, Mechanical Vibrations (6th edition, SI), Pearson, 2018 (ISBN 978-1-292-17860-8).

Dynamics 4
MECE10002
Semester 1, 2026-27

Classroom
Classroom 12, Alrick

Learning module
LM6

Course notes
Section 3.3

Slides
LM6.3 to LM6.18

Discussion thread
transverse_waves

Question 6.1

Using the Euler-Bernoulli equation show that a travelling flexural wave in a uniform beam will have a wave speed c that is not constant but depends on the wavelength. Assume a travelling wave deformation

$$w(x,t) = A_0 \sin\left[\frac{2\pi}{\lambda}(x-ct)\right]$$

and find how c depends on the wavelength λ. Is the result physically reasonable? Spoiler alert: it is not. What is the problem then?

Example solution

The Euler-Bernoulli equation is

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} = 0$$

Using the travelling wave assumption the deflection of the beam can be described by the function

$$w(x,t) = A_0 \sin\left[\frac{2\pi}{\lambda}(x-ct)\right]$$

Evaluating the second time and fourth space derivatives leads to

$$\frac{\partial^2 w}{\partial t^2} = -A_0\left(\frac{2\pi c}{\lambda}\right)^2 \sin\left[\frac{2\pi}{\lambda}(x-ct)\right]$$

and

$$\frac{\partial^4 w}{\partial x^4} = A_0\left(\frac{2\pi}{\lambda}\right)^4 \sin\left[\frac{2\pi}{\lambda}(x-ct)\right]$$

respectively. Substituting into the wave equation gives

$$EI\left(\frac{2\pi}{\lambda}\right)^4 - \rho A\left(\frac{2\pi c}{\lambda}\right)^2 = 0$$

or

$$c = \frac{2\pi}{\lambda}\sqrt{\frac{EI}{\rho A}}$$

This is a reasonable solution as long as the wavelength is not too small. For wavelengths close to zero the previous relation predicts that the wave speed goes to infinity, which is obviously not a good solution. This happens because the Euler-Bernoulli equation is only valid when the wavelength is much higher than the beam thickness.



Question 6.2

A uniform steel cantilever beam has the following dimensions: length $L = 0.4$ m, width $a = 35$ mm and depth $b = 8$ mm. Assume that $E = 207$ GPa and $\rho = 7840$ kg/m^3. Table 1 compares some natural frequency calculations (all values in Hz). Fill in the missing values.

Table 1: Frequencies calculated with different methods, in Hz (Question 6.2).

Method	Mode 1	Mode 2	Mode 3
Single mass model (try 1/3 of beam mass) at end of massless beam	?		
Rayleigh's method with parabolic assumed mode shape	?		
20 mass lumped mass model	41.45	259.06	723.53
FE using 4 elements	41.51	260.40	733.80
Euler-Bernoulli solution	?	?	?

Example solution

The second moment of area for a rectangular cross section is $I = ab^3/12$ and thus $EI = 309.1$ Nm^2. The total mass of the beam is $m = \rho abL = 0.87$ kg. For the **single degree-of-freedom approach** the cantilever is envisaged as an effective lumped mass (consider $m/3$) and the stiffness of the effective spring can be obtained directly from the static deflection of a cantilever under an applied load F. This deflection is[1]:

$$w = \frac{FL^3}{3EI}$$

Consequently, the spring stiffness is

$$k = \frac{F}{w} = \frac{3EI}{L^3} = 14490 \text{ N/m}$$

The natural frequency is then

$$\omega = \sqrt{\frac{k}{m/3}} = \sqrt{\frac{9EI}{mL^3}} = 223.5 \text{ rad/s} = 35.57 \text{ Hz}$$

For **Rayleigh's method** we can assume a parabolic mode shape, that is,

$$w(x,t) = \varphi(x)q(t) \quad \text{with} \quad \varphi(x) = \frac{x^2}{L^2}$$

The normalisation with L^2 ensures that the deflection at the free end of the beam is just the mode amplitude q. This is not strictly necessary but simplifies the calculations. As the beam is clamped the boundary condition for $x = 0$ is $\varphi(0) = 0$. The kinetic energy of the beam is

$$T = \frac{1}{2}\dot{q}^2 \rho A \int_0^L [\varphi(x)]^2 \, \mathrm{d}x = \frac{1}{2}\dot{q}^2 \frac{\rho AL}{5} = \frac{1}{2} m_{\text{eff}} \dot{q}^2$$

where $m_{\text{eff}} = m/5$. The potential energy of the beam is

$$V = \frac{1}{2} q^2 EI \int_0^L [\varphi(x)'']^2 \, \mathrm{d}x = \frac{1}{2} q^2 \frac{4EI}{L^3} = \frac{1}{2} k_{\text{eff}} q^2$$

where $k_{\text{eff}} = 4EI/L^3$. The natural frequency is then

$$\omega = \sqrt{\frac{k_{\text{eff}}}{m_{\text{eff}}}} = 333.2 \text{ rad/s} = 53.03 \text{ Hz}$$

[1] Think back to Structural Mechanics 2.

For the **Euler-Bernoulli approach** it is necessary to seek solutions of the differential equation

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} = 0$$

Using a standing wave solution the deflection of the beam is

$$w(x,t) = \varphi(x)\cos(\omega t)$$

The mode shape function $\varphi(x)$ fulfils the equation

$$\frac{\partial^4 \varphi}{\partial x^4} - \frac{\rho A\omega^2}{EI}\varphi = 0$$

with the general solution

$$\varphi(x) = C_1\cos(kx) + C_2\cosh(kx) + C_3\sin(kx) + C_4\sinh(kx) \quad \text{with} \quad k^4 = \frac{\rho A\omega^2}{EI}$$

The boundary conditions at the clamped end of the cantilever ($x = 0$) are $\varphi(0) = \varphi'(0) = 0$. This leads to $C_1 + C_2 = 0$ and $C_3 + C_4 = 0$. The boundary conditions at the free end of the cantilever ($x = L$) are $\varphi''(L) = \varphi'''(L) = 0$. Substituting $C_1 = -C_2 = C$ and $C_3 = -C_4 = D$, these conditions become

$$C\cos(kL) + C\cosh(kL) + D\sin(kL) + D\sinh(kL) = 0$$

$$C\sin(kL) - C\sinh(kL) - D\cos(kL) - D\cosh(kL) = 0$$

This system of equations on C and D only has non-trivial solutions if

$$\begin{vmatrix} \cos(kL) + \cosh(kL) & \sin(kL) + \sinh(kL) \\ \sin(kL) - \sinh(kL) & -\cos(kL) - \cosh(kL) \end{vmatrix} = 0$$

that solves to

$$1 + \cosh(kL)\cos(kL) = 0$$

$$\cos(kL) = -\frac{1}{\cosh(kL)}$$

The graphical solution for this equation is shown in Figure 1. The first root is located at $kL = 1.875$. As the inverse cosh gets closer to zero the remaining roots are approximately at $kL = (n + 1/2)\pi$, with $n = 1, \ldots$ Thus, the frequencies are

$$\omega_n = \sqrt{\frac{(kL)_n^4 EI}{mL^3}} = \left(n + \frac{1}{2}\right)^2 \pi^2 \sqrt{\frac{EI}{mL^3}}$$

that can be computed to the values listed in Table 2.

Table 2: Frequencies calculated with different methods, in Hz (Question 6.2).

Method	**Mode 1**	**Mode 2**	**Mode 3**
Single mass model (try 1/3 of beam mass) at end of massless beam	35.57		
Rayleigh's method with parabolic assumed mode shape	53.03		
20 mass lumped mass model	41.45	259.06	723.53
FE using 4 elements	41.51	260.40	733.80
Euler-Bernoulli solution	41.69	263.33	731.50

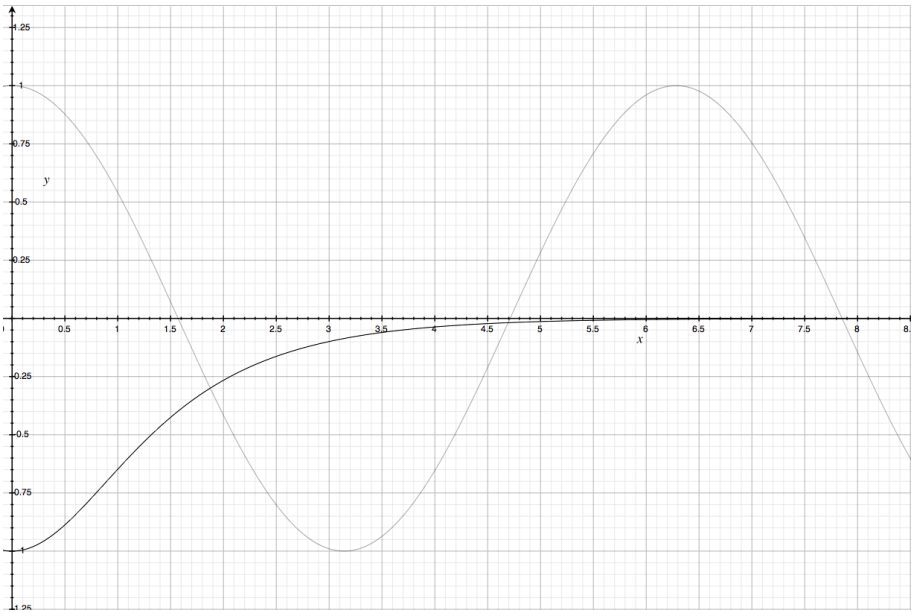


Figure 1: Graphical solution for Question 6.2.

Question 6.3

A uniform beam AB is simply supported at end A ($x = L$) while end B ($x = 0$) is rigidly clamped to a rigid wall. Apply these boundary conditions and show that the roots (kL) of the following equation

$$\tan(kL) = \tanh(kL)$$

can be used to determine the beam frequencies. Use a sketch plot to show where the roots are located. Determine the approximate value of the first root.

Example solution

The solutions for the Euler-Bernoulli equation

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} = 0$$

can be obtained in the form of standing waves $w(x,t) = \varphi(x)\cos(\omega t)$. Deriving $w(x,t)$ twice in time and four times in space, and inserting into the Euler-Bernoulli equation the mode shape function $\varphi(x)$ fulfils the equation

$$\frac{\partial^4 \varphi}{\partial x^4} - \frac{\rho A\omega^2}{EI}\varphi = 0$$

with the general solution

$$\varphi(x) = C_1\cos(kx) + C_2\cosh(kx) + C_3\sin(kx) + C_4\sinh(kx) \quad \text{with} \quad k^4 = \frac{\rho A\omega^2}{EI}.$$

The boundary conditions at the clamped end of the beam ($x = 0$) are $\varphi(0) = 0$ (zero deflection) and $\varphi'(0) = 0$ (zero slope). These two conditions lead to

$$C_1 + C_2 = 0$$

$$C_3 + C_4 = 0$$

hence $C_1 = -C_2 = C$ and $C_3 = -C_4 = D$. The boundary conditions at the simply supported end ($x = L$) are $\varphi(L) = 0$ (zero deflection) and $\varphi''(L) = 0$ (zero moment). These two conditions lead to

$$C\cos(kL) - C\cosh(kL) + D\sin(kL) - D\sinh(kL) = 0$$

$$-C\cos(kL) - C\cosh(kL) - D\sin(kL) - D\sinh(kL) = 0$$

This system of equations only has nontrivial solutions if its determinant is zero, that is,

$$\begin{vmatrix} \cos(kL) - \cosh(kL) & \sin(kL) - \sinh(kL) \\ -\cos(kL) - \cosh(kL) & -\sin(kL) - \sinh(kL) \end{vmatrix} = 0$$

This determinant is

$$2\sinh(kL)\cos(kL) - 2\cosh(kL)\sin(kL) = 0$$

After rearrangement this equation becomes

$$\tanh(kL) = \tan(kL)$$

The intersections of functions $\tanh(kL)$ and $\tan(kL)$ are in a regime where the hyperbolic tangent is close to 1. So, the approximate solution is $(kL) \approx \pi(n + 1/4)$ with $n = 1, \ldots$
Solving for the natural frequencies finally leads to

$$\omega = \left[\frac{\pi}{L}\left(n + \frac{1}{4}\right)\right]^2 \sqrt{\frac{EI}{\rho A}}$$

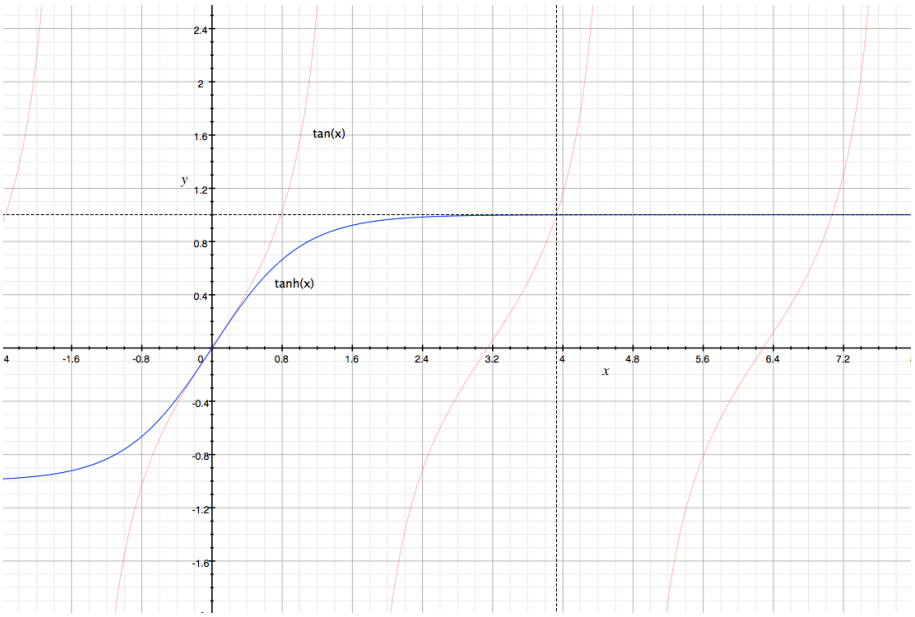


Figure 2: Graphical solution for Question 6.3.



Question 6.4

A long uniform beam is suspended from a frictionless pivot and hangs vertically. If disturbed the beam will behave like a pendulum. The frequency of the pendulum can be calculated assuming the beam is rigid. However, the beam is not rigid and, if struck sideways, will oscillate at many frequencies as well as swinging as a pendulum.
Use Euler-Bernoulli's equation to find a frequency equation for the pendulum beam. Assume that gravity has a negligible effect on this.

Example solution

As in the previous question, the solutions for the Euler-Bernoulli equation

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} = 0$$

can be obtained in the form of a standing wave $w(x,t) = \varphi(x)\cos(\omega t)$. Deriving $w(x,t)$ twice in time and four times in space, and inserting into the Euler-Bernoulli equation above, the mode shape function $\varphi(x)$ fulfils the equation

$$\frac{\partial^4 \varphi}{\partial x^4} - \frac{\rho A\omega^2}{EI}\varphi = 0,$$

with the general solution

$$\varphi(x) = C_1\cos(kx) + C_2\cosh(kx) + C_3\sin(kx) + C_4\sinh(kx) \quad \text{with} \quad k^4 = \frac{\rho A\omega^2}{EI}.$$

The boundary conditions at the simply supported top end of the beam ($x = 0$) are $\varphi(0) = 0$ (zero deflection) and $\varphi''(0) = 0$ (zero moment). From these conditions $C_1 = C_2 = 0$ and the shape function reduces to

$$\varphi(x) = C_3\sin(kx) + C_4\sinh(kx).$$

For free end of the beam ($x = L$) moment and stress are zero so $\varphi''(L) = \varphi'''(L) = 0$. From these conditions

$$-C_3\sin(kL) + C_4\sinh(kL) = 0$$
$$-C_3\cos(kL) + C_4\cosh(kL) = 0.$$

The resulting solvability condition is

$$\sinh(kL)\cos(kL) - \cosh(kL)\sin(kL) = 0,$$

which is the same as in the previous question. So, after rearrangement, this equation becomes

$$\tanh(kL) = \tan(kL).$$

Once again, the intersections of functions $\tanh(kL)$ and $\tan(kL)$ are in a regime where the hyperbolic tangent is close to 1. So, the approximate solution is $(kL) \approx \pi(n+1/4)$ with $n = 1, \ldots$ Considering the definition above for k the natural frequencies can be determined to be

$$\omega = \left[\frac{\pi}{L}\left(n + \frac{1}{4}\right)\right]^2\sqrt{\frac{EI}{\rho A}}$$



Question 6.5

If a cantilever beam of length L has an additional mass M attached at the free end what boundary conditions would apply to the Euler-Bernoulli solution? Hint: the shear force at the beam end would not be zero, but would be related to the transverse inertia force of the mass moving with acceleration of the beam end.

Example solution

Assuming that the end with the attached mass is at $x = L$ the mass experiences an inertial force of magnitude

$$F = M\ddot{w}(L)$$

This (external) inertial force must be balanced by the (internal) shear force F_s at the end of the beam. This is equivalent to saying that the magnitude of the internal and external forces needs to be the same, that is, $|F| - |F_\mathrm{s}| = 0$

$$|F_\mathrm{s}| = A|\tau(L)| = \left|-EI\frac{\partial^3 w(L,t)}{\partial x^3}\right| = EI\frac{\partial^3 w(L,t)}{\partial x^3}$$

$$|F| - |F_\mathrm{s}| = 0 \quad \Leftrightarrow \quad M\ddot{w}(L) - EI\frac{\partial^3 w(L,t)}{\partial x^3} = 0$$

Hence, the boundary conditions at $x = L$ are

$$\left.\frac{\partial^2 w}{\partial x^2}\right|_{x=L} = 0 \quad \text{and} \quad EI\left.\frac{\partial^3 w}{\partial x^3}\right|_{x=L} = M\ddot{w}(L)$$

Question 6.6

The structure shown in Figure 3 is a circular cross-section (radius r) beam/bar clamped at one end and free at the other. Assume a travelling wave approach and obtain the ratio between the transverse and longitudinal (axial) wave speeds for the beam/bar.

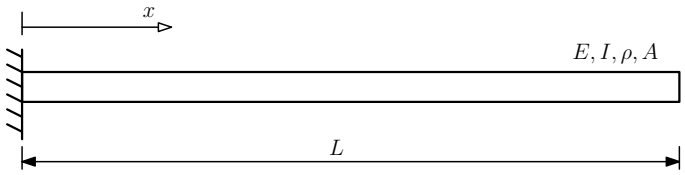


Figure 3: Clamped beam/bar for Question 6.6.

Example solution

The transverse wave is described by the Euler-Bernoulli differential equation

$$\rho A \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^4 w}{\partial x^4} = 0$$

Assuming a travelling wave approach, the solution for the wave equation is

$$w(x,t) = A \sin\left[\frac{2\pi}{\lambda_\mathrm{t}}\left(x - c_\mathrm{t} t\right)\right]$$

As the beam is clamped at $x = 0$ and free at $x = L$, the boundary conditions for the beam are $w(0,t) = w'(0,t) = w''(L,t) = w'''(L,t) = 0$. Differentiating $w(x,t)$ and inserting into the Euler-Bernoulli equation leads to

$$\left(\frac{2\pi}{\lambda_\mathrm{t}}\right)^2 = \frac{\rho A c_\mathrm{t}^2}{EI} \quad \Rightarrow \quad c_\mathrm{t} = \frac{2\pi}{\lambda_\mathrm{t}}\sqrt{\frac{EI}{\rho A}}$$

The axial (longitudinal) wave is described by the following differential wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho}\frac{\partial^2 u}{\partial x^2}$$

Assuming once again a travelling wave approach, the solution for the wave equation is

$$u(x,t) = B \sin\left[\frac{2\pi}{\lambda_\mathrm{l}}\left(x - c_\mathrm{l} t\right)\right]$$

As the bar is fixed for $x = 0$ and free for $x = L$ the boundary conditions for the bar are $u(0,t) = u'(L,t) = 0$. Differentiating $u(x,t)$ and inserting into the wave equation yields

$$c_\mathrm{l}^2 = \frac{E}{\rho}$$

The wave speed ratio becomes

$$\frac{c_\mathrm{t}^2}{c_\mathrm{l}^2} = \frac{EI}{\rho A}\left(\frac{2\pi}{\lambda_\mathrm{t}}\right)^2 \frac{\rho}{E} \quad \Rightarrow \quad \frac{c_\mathrm{t}}{c_\mathrm{l}} = \frac{2\pi}{\lambda_\mathrm{t}}\sqrt{\frac{I}{A}}$$

The geometrical properties for a circular cross section beam/bar are

$$I = \frac{\pi r^4}{4} \quad \text{and} \quad A = \pi r^2$$

Consequently,

$$\sqrt{\frac{I}{A}} = \frac{r}{2}$$

and the ratio of the wave speeds is

$$\frac{c_\mathrm{t}}{c_\mathrm{l}} = \frac{\pi r}{\lambda_\mathrm{t}}$$

Question 6.7

The beams shown in Figures 4 (a) to (d) are clamped at $x = 0$ and have different boundary conditions at $x = L$. The beams have circular cross section ($r = 10$ mm), $L = 2$ m and are made from steel with $E = 200$ GPa and $\rho = 8000$ kg/m^3. Consider $K = 1.0 \times 10^6$ N/m and a standing wave solution.

a) State the boundary conditions for all the beams in figures 4(a) to (d).
b) Obtain the first three wave frequencies for the beams in figures 4(a) and (b).
c) Compare and discuss the obtained frequencies.

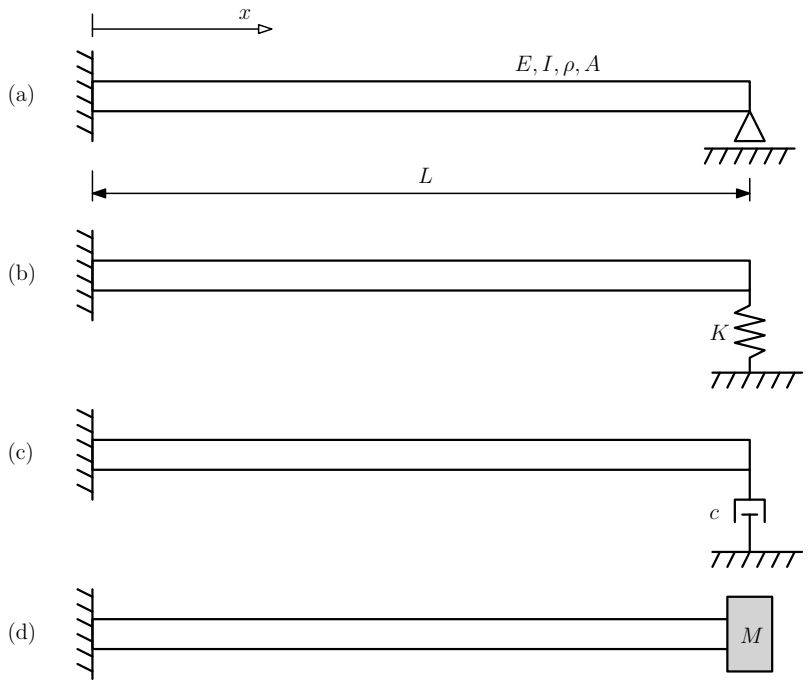


Figure 4: Clamped beams with different boundary conditions for Question 6.7.

Example solution

We start by considering a standing wave solution $w(x,t) = \varphi(x)\theta(t) = \varphi(x)\cos(\omega t)$ where the shape function is

$$\varphi(x) = C_1\cos(kx) + C_2\sin(kx) + C_3\cosh(kx) + C_4\sinh(kx)$$

with

$$k^4 = \frac{\rho A\omega^2}{EI}$$

The beams are clamped at $x = 0$ meaning that the corresponding boundary conditions are $\varphi(0) = \varphi'(0) = 0$ for all the situations shown. The beams are also all moment free at $x = L$. Consequently, the boundary condition $\varphi''(L) = 0$ is also common to all the situations shown.
The beam in Figure 4(a) has a pinned support at $x = L$, that is, there is also no vertical displacement at this end. The corresponding boundary conditions are then $\varphi(L) = \varphi''(L) = 0$.
The beam in Figure 4(b) has a spring with stiffness K attached at $x = L$. The corresponding boundary condition is due to the spring force and is (see more detail in the solution to Question 6.5)

$$EI\left.\frac{\partial^3 w(x,t)}{\partial x^3}\right|_{x=L} = Kw(L)$$

The beam in Figure 4(c) has a viscous damper with damping constant c attached at $x = L$. The corresponding boundary condition is due to the force of the damper and is

$$EI\left.\frac{\partial^3 w(x,t)}{\partial x^3}\right|_{x=L} = c\dot{w}(L)$$

The beam in Figure 4(d) has a mass m attached at $x = L$. The corresponding boundary condition is due to the inertial force of the mass and is

$$EI\left.\frac{\partial^3 w(x,t)}{\partial x^3}\right|_{x=L} = m\ddot{w}(L)$$

The first three boundary conditions are common to the beams in Figures 4(a) and (b). Inserting these three boundary conditions in the $w(x,t)$ solution leads to

$$C_1 + C_3 = 0$$

$$C_2 + C_4 = 0$$

$$-C_1\cos(kL) - C_2\sin(kL) + C_3\cosh(kL) + C_4\sinh(kL) = 0$$

For the beam in Figure 4(a) the additional boundary condition yields

$$C_1\cos(kL) + C_2\sin(kL) + C_3\cosh(kL) + C_4\sinh(kL) = 0$$

These four equations constitute a linear system of equations on C_1, C_2, C_3 and C_4. For this system of equations to have a non-trivial solution the determinant of the system's matrix must be zero, that is

$$\begin{vmatrix} \cos(kL) & \sin(kL) & \cosh(kL) & \sinh(kL) \\ -\cos(kL) & -\sin(kL) & \cosh(kL) & \sinh(kL) \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{vmatrix} = 0$$

Solving this determinant leads to the following relation

$$\sin(kL)\cosh(kL) = \cos(kL)\sinh(kL)$$

with the solutions listed in Table 3 and

$$\omega = (kL)^2\sqrt{\frac{EI}{mL^3}}$$

A similar procedure can be followed for the beam in Figure 4(b). The first three boundary conditions (and equations) are exactly the same as for the beam in Figure 4(a). The third derivative of the shape function is

$$\varphi'''(x) = k^3\left[C_1\sin(kx) - C_2\cos(kx) + C_3\sinh(kx) + C_4\cosh(kx)\right]$$

and thus the fourth boundary condition becomes

$$\left[C_1\sin(kL) - C_2\cos(kL) + C_3\sinh(kL) + C_4\cosh(kL)\right] =$$

$$= A\left[C_1\cos(kL) + C_2\sin(kL) + C_3\cosh(kL) + C_4\sinh(kL)\right]$$

with

$$A = \frac{K}{EIk^3}$$

These four equations constitute once again a linear system of equations on C_1, C_2, C_3 and C_4. For this system of equations to have a non-static solution the determinant of the system's matrix must be zero, that is

$$\begin{vmatrix} \sin(kL) - A\cos(kL) & -A\sin(kL) - \cos(kL) & \sinh(kL) - A\cosh(kL) & \cosh(kL) - A\sinh(kL) \\ -\cos(kL) & -\sin(kL) & \cosh(kL) & \sinh(kL) \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{vmatrix} = 0$$

Solving this determinant leads to

$$(kL)^3\left[1 + \cos(kL)\cosh(kL)\right] + \frac{KL^3}{EI}\left[\sin(kL)\cosh(kL) - \cos(kL)\sinh(kL)\right] = 0$$

with the solutions listed in Table 3 and, once again,

$$\omega = (kL)^2\sqrt{\frac{EI}{mL^3}}$$

A different approach to the solution of this question would be to insert the relations

$$C_1 + C_3 = 0 \quad \text{and} \quad C_2 + C_4 = 0$$

at the beginning of the solution and dealing with 2×2 determinants (instead of 4×4). The boundary conditions for the beams in Figures 4(a) and (b) are somewhat similar. The simple support at $x = L$ in Figure 4(a) can be seen as an infinite stiffness spring, as in the beam in Figure 4(b). Consequently, the natural frequencies for the beam with the attached spring are expected to be slightly lower, as can be seen from the obtained values.



Table 3: First three wave frequencies for the beams in Figures 4 (a) and (b) for Question 6.7.

	(a)			(b)		
n Frequency	kL	ω_n [rad]	ω_n [Hz]	kL (b)	ω_n [rad]	ω_n [Hz]
1	3.9266	96.36	15.34	3.9210	96.09	15.29
2	7.0686	312.28	49.70	7.0331	309.15	49.20
3	10.2102	651.55	103.70	10.0981	637.32	101.43



Question 6.8

A uniform elastic beam of length L, cross-section area A and second moment of area I has density ρ and elastic modulus E. Find the mode shapes and modal frequencies if the beam is simply supported.

Example solution

In this example we have to evaluate the vibration modes of a beam that is simply supported. We assume the left end to be at $x = 0$ and the right end at $x = L$. Consequently, the boundary conditions are

$$w(x = 0, t) = w''(x = 0, t) = 0$$
$$w(x = L, t) = w''(x = L, t) = 0$$

We assume the solution in the form $w(x, t) = \varphi(x)\cos(\omega t)$. Inserting in the Euler-Bernoulli equation then gives

$$\left[EI\frac{\partial^4\varphi}{\partial x^4}(x) - A\rho\omega^2\varphi(x)\right]\cos(\omega t) = 0$$

and the equation for the space-dependent mode shape function becomes

$$\frac{\partial^4\varphi}{\partial x^4}(x) - k^4\varphi(x) = 0 \qquad \text{with} \qquad k^4 = \frac{A\rho\omega^2}{EI}$$

The general solution of this equation and the derivatives of the general solution are

$$\varphi(x) = C_1\cos(kx) + C_2\sin(kx) + C_3\cosh(kx) + C_4\sinh(kx)$$
$$\varphi'(x) = k\left[-C_1\sin(kx) + C_2\cos(kx) + C_3\sinh(kx) + C_4\cosh(kx)\right]$$
$$\varphi''(x) = k^2\left[-C_1\cos(kx) - C_2\sin(kx) + C_3\cosh(kx) + C_4\sinh(kx)\right]$$

The boundary conditions at the simply supported beam ends are

$$\varphi(x = 0) = \varphi''(x = 0) = 0$$
$$\varphi(x = L) = \varphi''(x = L) = 0$$

Inserting the derivatives of the general solution and its derivatives, the boundary conditions at $x = 0$ yield

$$\varphi(0) = C_1 + C_3 = 0$$
$$\varphi''(0) = k^2(-C_1 + C_3) = 0$$

Solving for the unknown constants we get $C_1 = C_3 = 0$ and the solution to the wave equation becomes

$$\varphi(x) = C_2\sin(kx) + C_4\sinh(kx)$$
$$\varphi''(x) = k^2\left[-C_2\sin(kx) + C_4\sinh(kx)\right]$$

The boundary conditions at $x = L$ produce the following system of equations for the two unknown constants C_2 and C_4

$$C_2\sin(kL) + C_4\sinh(kL) = 0$$
$$-C_2\sin(kL) + C_4\sinh(kL) = 0$$

For this system of equations to have a nontrivial solution the determinant of the matrix of coefficients must be zero, that is,

$$2\sin(kL)\sinh(kL) = 0$$

Since the $\sinh$ function only has one zero at $x = 0$, the zeros of this equation are those of the sine function,

$$\sin(kL) = 0 \qquad \text{with} \qquad kL = n\pi$$

It also follows that $C_4 = 0$, so the solutions are pure sines. It can be seen that n simply counts the number of half waves within the length of the bar. The corresponding oscillation frequencies can be obtained from

$$k^4 = \frac{A\rho\omega^2}{EI} \qquad \text{and} \qquad \omega^2 = \frac{EIk^4}{A\rho} = \frac{EI(kL)^4}{A\rho L^4} = \frac{\pi^4 EI}{A\rho L^4}n^4$$

$$\Rightarrow \qquad \omega = \left(\frac{\pi n}{L}\right)^2\sqrt{\frac{EI}{A\rho}} \quad \text{for} \quad n = 1, 2, \ldots$$

Dynamics 4

Exercise Sheet 7

Notes for this Exercise Sheet This exercise sheet covers the following topics: (i) free vibration of beams and flexural wave propagation in continuous uniform beams; (ii) beam shape functions and boundary conditions; (iii) Rayleigh-Ritz mode approximation; (iv) impact on slender structural elements.

Dynamics 4
MECE10002
Semester 1, 2026-27

Classroom
Classroom 12, Alrick

Learning module
LM7

Course notes
Section 4.1
Section 4.2

Slides
LM7.3 to LM7.32

Discussion thread
impact_on_structures

Question 7.1

A uniform and homogeneous beam of length L, with Young's modulus E and cross-sectional second moment of area I has both ends fully clamped. This beam satisfies the equation of motion

$$0 = \frac{\partial^2 w}{\partial t^2} + \gamma^2 \frac{\partial^4 w}{\partial x^4} \quad \text{with} \quad \gamma^2 = \frac{EI}{\rho A}$$

a) What are the boundary conditions for the beam?
b) Suggest adequate shape functions $X_1(x)$ and $X_2(x)$ for the first (symmetric) and second (antisymmetric) mode shapes of the beam.
c) Estimate the frequencies of the first symmetric and antisymmetric modes.

Example solution

As the beam is clamped at both ends the boundary conditions are

$$w(0,t) = 0 \quad \text{and} \quad w(L,t) = 0$$

corresponding to zero displacement (deflection) at both ends, and

$$\left.\frac{\partial w(x,t)}{\partial x}\right|_{x=0} = 0 \quad \text{and} \quad \left.\frac{\partial w(x,t)}{\partial x}\right|_{x=L} = 0$$

corresponding to zero rotation at both ends. Assuming the deflection can be described as a standing wave, $w(x,t) = X(x)\theta(t) = X(x)\cos(\omega t)$ the above boundary conditions can be redefined as $X(0,t) = 0$ and $X(L,t) = 0$ for the transverse displacement and

$$\left.\frac{\partial X(x)}{\partial x}\right|_{x=0} = 0 \quad \text{and} \quad \left.\frac{\partial X(x)}{\partial x}\right|_{x=L} = 0$$

for the beam end rotation. Considering the above boundary conditions, possible shape functions for the first (symmetric) and second (antisymmetric) mode shapes are

$$X_1(x) = 1 - \cos\left(\frac{2\pi x}{L}\right) \qquad \text{and} \qquad X_2(x) = \sin\left(\frac{2\pi x}{L}\right) - \frac{1}{2}\sin\left(\frac{4\pi x}{L}\right)$$

Differentiating $w(x,t) = X(x)\cos(\omega t)$ twice in time and four times in space, and inserting into the Euler-Bernoulli equation, the mode shape function $X(x)$ fulfils the equation

$$\frac{\mathrm{d}^4 X(x)}{\mathrm{d}x^4} - \frac{\rho A \omega^2}{EI} X(x) = 0$$

Using Rayleigh-Ritz's approximation to find an equivalent single degree-of-freedom system, the natural frequencies ω_i can be calculated as

$$\omega_i = \sqrt{\frac{\bar{K}_i}{\bar{M}_i}}$$

where $\bar{K}_i$ and $\bar{M}_i$ are the effective (equivalent) stiffness and mass, respectively, given by

$$\bar{K}_i = \int_0^L EI \left(\frac{\mathrm{d}^2 X_i}{\mathrm{d}x^2}\right)^2 \mathrm{d}x \qquad \text{and} \qquad \bar{M}_i = \int_0^L \rho A X_i^2 \mathrm{d}x$$



For the first natural frequency, we use the first shape function $X_1(x)$. The effective stiffness becomes

$$\bar{K}_1 = EI \int_0^L \left(\frac{2\pi}{L}\right)^4 \cos^2\left(\frac{2\pi x}{L}\right) =$$

$$= EI \left(\frac{2\pi}{L}\right)^4 \left[\frac{x}{2} + \frac{L}{4\pi}\sin\left(\frac{2\pi x}{L}\right)\cos\left(\frac{2\pi x}{L}\right)\right]_0^L = 779\frac{EI}{L^3}$$

and the effective mass (inertia) is

$$\bar{M}_1 = \rho A \int_0^L \left[1 - \cos\left(\frac{2\pi x}{L}\right)\right]^2 \mathrm{d}x =$$

$$= \rho A \left[x + \frac{x}{2} + \frac{L}{4\pi}\sin\left(\frac{2\pi x}{L}\right)\cos\left(\frac{2\pi x}{L}\right) - \frac{L}{\pi}\sin\left(\frac{2\pi x}{L}\right)\right]_0^L = \frac{3\rho AL}{2}$$

The natural frequency for the first (symmetrical) mode shape of the beam is

$$\omega_1 = \sqrt{\frac{\bar{K}_1}{\bar{M}_1}} = \sqrt{779\frac{EI}{L^3}\frac{2}{3\rho AL}} = 22.8\frac{\gamma}{L^2}$$

Proceeding in a similar way for the natural frequency of the second (antisymmetrical) mode shape leads to

$$\omega_2 = \sqrt{\frac{\bar{K}_2}{\bar{M}_2}} = 78.9\frac{\gamma}{L^2}$$

Although the generic approach is correct there is something wrong with this example solution. Can you identify what the problem is?



Question 7.2

A uniform beam of length L and mass $M = \rho AL$ is fully clamped at both ends. For the mode shape approximation

$$X(x) = \frac{1}{2}\left[1 - \cos\left(\frac{2\pi x}{L}\right)\right]$$

find the equivalent mass $\bar{M}$ and modal frequency ω for free vibration of the beam.

Example solution

See example solution of previous question.

Question 7.3

A uniform elastic beam of length L, area of cross-section A, elastic modulus E and second moment of area I is fully clamped at both ends. The beam is initially at rest before it is impacted at mid-span by a rigid missile of mass m, travelling with speed v_0 in a direction perpendicular to the axis of the beam, as shown in Figure 1.

a) Find the response of the beam for a mass ratio $m/M = 0.1$.

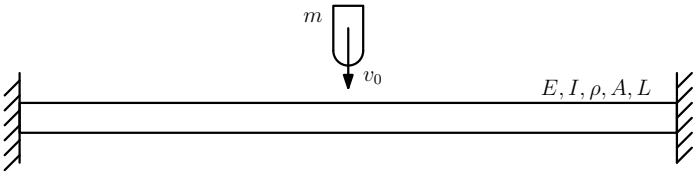


Figure 1: Impact of a rigid missile on a uniform beam (Question 7.3).

Example solution

Assuming the following symmetric and antisymmetric mode approximations:

$$X_1 = 1 - \cos\left(\frac{2\pi x}{L}\right) \qquad \text{and} \qquad X_2 = \sin\left(\frac{2\pi x}{L}\right) - \frac{1}{2}\sin\left(\frac{4\pi x}{L}\right)$$

and using Rayleigh's method approximation, the modal frequencies are given by the relation

$$\omega_i^2 = \int_0^L EI\left(\frac{\mathrm{d}^2 X_i}{\mathrm{d}x^2}\right)^2 \mathrm{d}x \Big/ \int_0^L \rho A X_i^2 \mathrm{d}x$$

Combining the mode approximations and the above relation for ω_i^2, and integrating the result gives

$$\omega_1 = 22.80\frac{\gamma}{L^2} \quad \text{and} \quad \omega_2 = 78.98\frac{\gamma}{L^2}$$

where $\gamma^2 = EI/(\rho A)$. The modified modal frequencies are

$$\bar{\omega}_i^2 = w_i^2 \Big/ \left(1 + \frac{m|X_i(x_0)|}{\rho A \int_0^L X_i^2 \mathrm{d}x}\right)$$

Substituting the frequencies of the first symmetric and antisymmetric modes of the beam leads to the modified modal frequencies

$$\bar{\omega}_1 = 22.80\frac{\gamma}{L^2} \Big/ \sqrt{1 + \frac{2m|X_1(x_0)|}{3M}}$$

$$\bar{\omega}_2 = 78.98\frac{\gamma}{L^2} \Big/ \sqrt{1 + \frac{8m|X_2(x_0)|}{5M}}$$

The amplitude coefficients are

$$A_i = \frac{m v_0 X_i(x_0)}{\bar{\omega}_i \rho A \int_0^L X_i^2 \mathrm{d}x}$$

Introducing the frequencies of the symmetric and antisymmetric modes of the beam we get

$$A_1 = \frac{L^2 v_0}{22.8\gamma} \frac{2mX_1(x_0)}{3M} \sqrt{1 + \frac{2m|X_1(x_0)|}{3M}}$$

$$A_2 = \frac{L^2 v_0}{78.98\gamma} \frac{8mX_2(x_0)}{5M} \sqrt{1 + \frac{8m|X_2(x_0)|}{5M}}$$

Note that the shape functions we are using in this solution are not normalised and as such the resulting amplitudes will be scaled amplitudes and not the actual beam displacements. The frequencies and amplitudes calculated for $x_0 = L/2$ and for the mass ratio $m/M = 0.1$ are listed in Table 1.



Table 1: Modified frequencies and amplitudes for the impacted beam in Figure 1.

m/M	$\bar{\omega}_1 L^2/\gamma$	$A_1\gamma/(L^2 v_0)$	$\bar{\omega}_2 L^2/\gamma$	$A_2\gamma/(L^2 v_0)$
0.1	21.4	0.006	79.0	0.0

Question 7.4

A uniform elastic beam of length L, area of cross-section A and second moment of area I has density ρ and elastic modulus E and is simply supported. The beam is at rest when it is impacted at mid-span by a rigid missile of mass m and velocity v_0 (see Figure 2). For shape functions

$$X_j(x) = \sin(k_j x) \quad \text{with} \quad k_j = \frac{j\pi}{L}$$

find the equivalent mass $\bar{M}_j$, the equivalent stiffness $\bar{K}_j$ and the amplitude A_j of the response of the beam as a function of the mass ratio $m/M = \beta$.

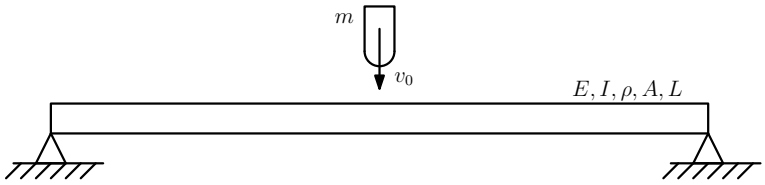


Figure 2: Impact of a rigid missile on a uniform simply supported beam (Question 7.4).

Example solution

The equivalent masses (inertias) are

$$\bar{M}_j = \rho A \int_0^L X_j^2 \mathrm{d}x = \rho A \frac{L}{2} = \frac{M}{2}$$

and the equivalent stiffnesses are

$$\bar{K}_j = EI \int_0^L \left(\frac{\mathrm{d}^2 X_j}{\mathrm{d}x^2} \right)^2 \mathrm{d}x = \frac{(j\pi)^4}{2} \frac{EI}{L^3}$$

The natural frequencies are then

$$\omega_j^2 = \frac{\bar{K}_j}{\bar{M}_j} = \frac{(j\pi)^4}{2} \frac{EI}{L^3} \frac{2}{\rho A L}$$

To determine the amplitudes A_j it is first necessary to calculate the modified frequencies, which are

$$\bar{\omega}_j^2 = \omega_j^2 \Big/ \left[1 + \frac{m|X_j(x_0)|}{\rho A \int_0^L X_j^2 \mathrm{d}x} \right] = \omega_j^2 \Big/ \left[1 + \frac{2m \sin(k_j x_0)}{\rho A L} \right] = \omega_j^2 \Big/ \left[1 + 2\sin(k_j x_0)\beta \right]$$

The amplitudes then become

$$A_j = \frac{m v_o X_j(x_0)}{\bar{\omega}_j \rho A \int_0^L X_j^2 \mathrm{d}x} = \frac{2m v_o X_j(x_0)}{\bar{\omega}_j \rho A L} = \frac{2 v_o X_j(x_0)}{\bar{\omega}_j} \beta$$



Question 7.5

A uniform elastic beam of length L, area of cross-section A, elastic modulus E and second moment of area I is simply supported. The beam is initially at rest before it is impacted at $x = L/3$ by a rigid missile of mass m, travelling with speed v_0 in a direction perpendicular to the axis of the beam.

a) Find the response of the beam if the missile weighs 10% of the weight of the beam.
b) Find the response of the beam if the missile weighs 1% of the weight of the beam.

Example solution

See example solution of previous question, with $x_0 = L/3$ and $\beta = 0.1$:

$$\omega_1^2 = \left(\frac{\pi}{L}\right)^4 \frac{EI}{\rho A}, \quad \bar{\omega}_1^2 = \left(\frac{\pi}{L}\right)^4 \frac{EI}{\rho A} \Big/ \left[1 + 0.2 \sin\left(\frac{\pi}{3}\right)\right] = 9.11 \frac{\gamma}{L^2},$$

$$\text{and} \quad A_1 = \frac{0.2 v_0 L^2}{9.11\gamma} X_1\left(\frac{L}{3}\right) = 0.019 \frac{L^2}{\gamma}$$

$$\omega_2^2 = \left(\frac{2\pi}{L}\right)^4 \frac{EI}{\rho A}, \quad \bar{\omega}_2^2 = \left(\frac{2\pi}{L}\right)^4 \frac{EI}{\rho A} \Big/ \left[1 + 0.2 \sin\left(\frac{2\pi}{3}\right)\right] = 36.45 \frac{\gamma}{L^2},$$

$$\text{and} \quad A_2 = \frac{0.2 v_0 L^2}{36.45\gamma} X_1\left(\frac{L}{3}\right) = 0.0048 \frac{L^2}{\gamma}$$

See example solution of previous question, with $x_0 = L/3$ and $\beta = 0.01$:

$$\omega_1^2 = \left(\frac{\pi}{L}\right)^4 \frac{EI}{\rho A}, \quad \bar{\omega}_1^2 = \left(\frac{\pi}{L}\right)^4 \frac{EI}{\rho A} \Big/ \left[1 + 0.02 \sin\left(\frac{\pi}{3}\right)\right] = 9.78 \frac{\gamma}{L^2},$$

$$\text{and} \quad A_1 = \frac{0.02 v_0 L^2}{9.78\gamma} X_1\left(\frac{L}{3}\right) = 0.0018 \frac{L^2}{\gamma}$$

$$\omega_2^2 = \left(\frac{2\pi}{L}\right)^4 \frac{EI}{\rho A}, \quad \bar{\omega}_2^2 = \left(\frac{2\pi}{L}\right)^4 \frac{EI}{\rho A} \Big/ \left[1 + 0.02 \sin\left(\frac{2\pi}{3}\right)\right] = 39.14 \frac{\gamma}{L^2},$$

$$\text{and} \quad A_2 = \frac{0.02 v_0 L^2}{39.14\gamma} X_1\left(\frac{L}{3}\right) = 0.00044 \frac{L^2}{\gamma}$$

Question 7.6

The thin metal blade in Figure 3 is made of brass (elastic modulus $E = 97$ GPa, density $\rho = 8500$ kg/m^3) and used as a tuning device for musical instruments.

a) Find the thickness of the blade t that will ensure the blade has a natural frequency that's equal to a key of A_4, which is 440 Hz.
b) If the blade is made to vibrate by impacting it with a metal sphere of mass $m = 1$ g with speed $v_0 = 1.4$ m/s at $x_0 = 2L/3$, how much out of tune will the blade resonate? Give your answer in percentage relative to the key of A_4 (440 Hz).
c) Find the corresponding amplitude of vibration of the tuning blade.

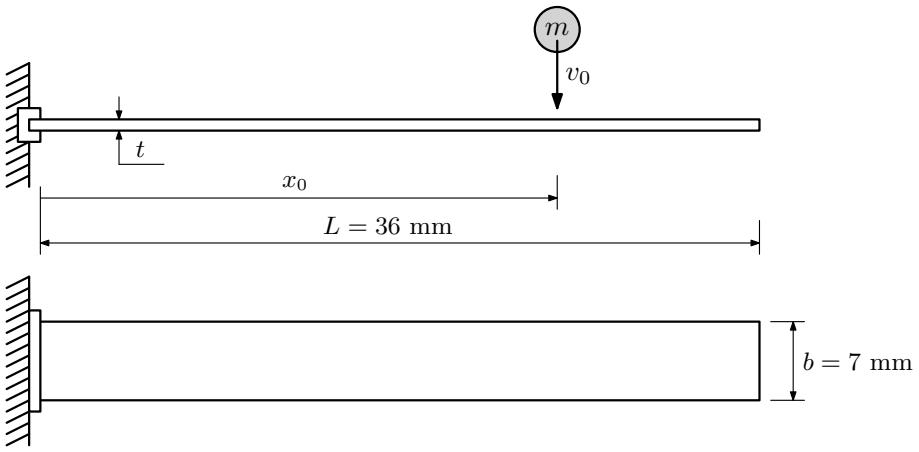


Figure 3: Tuning metallic blade for Question 7.6.

Example solution

To find the thickness of the blade t that ensures the blade has a natural frequency of 440 Hz we need to find the second moment of area of the blade I. For this solution we will assume the following normalised shape function for the beam

$$X(x) = \left(\frac{x}{L}\right)^2$$

which verifies the boundary conditions at the clamped end: $X(x=0) = 0$ and $X'(x=0) = 0$. The natural frequency of the beam is

$$\omega_1^2 = \int_0^L EI \left(\frac{\mathrm{d}^2 X}{\mathrm{d}x^2}\right)^2 \mathrm{d}x \Big/ \int_0^L \rho A X^2 \mathrm{d}x$$

where, using the chosen shape function,

$$\left(\frac{\mathrm{d}^2 X}{\mathrm{d}x^2}\right)^2 = \frac{4}{L^4} \qquad \text{and} \qquad X^2 = \frac{x^4}{L^4}$$

The natural frequency becomes

$$\omega_1^2 = EI \int_0^L \frac{4}{L^4} \mathrm{d}x \Big/ \left(\rho A \int_0^L \frac{x^4}{L^4} \mathrm{d}x\right)$$

which can be further simplified to give

$$\omega_1^2 = \frac{20EI}{\rho A L}\left(\frac{1}{L^3}\right)$$

As the cross-section of the beam is rectangular, $I = bt^3/12$ and $A = bt$, leading to

$$\omega_1^2 = \frac{5}{3}\frac{Et^2}{\rho L^4}$$

This can be rearranged to give

$$t = \omega_1 L^2 \sqrt{\frac{3\rho}{5E}}$$

Substituting all known values leads to

$$t = 2\pi \times 440 \times \left(36 \times 10^{-3}\right)^2 \sqrt{\frac{3 \times 8500}{5 \times 97 \times 10^9}} = 0.8 \times 10^{-3} = 0.8 \text{ mm}$$

In this part of the question we find the modified frequency after an impact with a metal sphere of mass $m = 1$ g with speed $v_0 = 1.4$ m/s at $x_0 = 2L/3$, which is given by

$$\bar{\omega}_1^2 = \omega_1^2 \Big/ \left(1 + \frac{mX(x_0)}{\rho A \int_0^L X^2 \mathrm{d}x}\right)$$

Substituting the known values

$$\bar{\omega}_1 = 2\pi \times 440 \Big/ \sqrt{1 + \frac{1 \times 10^{-3} \times X(2L/3)}{8500 \times 7 \times 10^{-3} \times 0.8 \times 10^{-3} \int_0^L X^2 \mathrm{d}x}}$$

and knowing that

$$X\left(\frac{2L}{3}\right) = \frac{4}{9} \qquad \text{and} \qquad \int_0^L X^2 \mathrm{d}x = \frac{L}{5}$$

the modified frequency becomes $\bar{\omega}_1 = 1818.8$ rad/s, which corresponds to a 34.2% difference from the natural frequency of the tuning blade. In the impact situation described above, the amplitude of vibration of the tuning blade can be calculated with the following relation

$$A_1 = \frac{mv_0 X(x_0)}{\bar{\omega}_1 \rho A \int_0^L X^2 \mathrm{d}x} = \frac{20 m v_0}{9 \bar{\omega}_1 \rho A L}$$

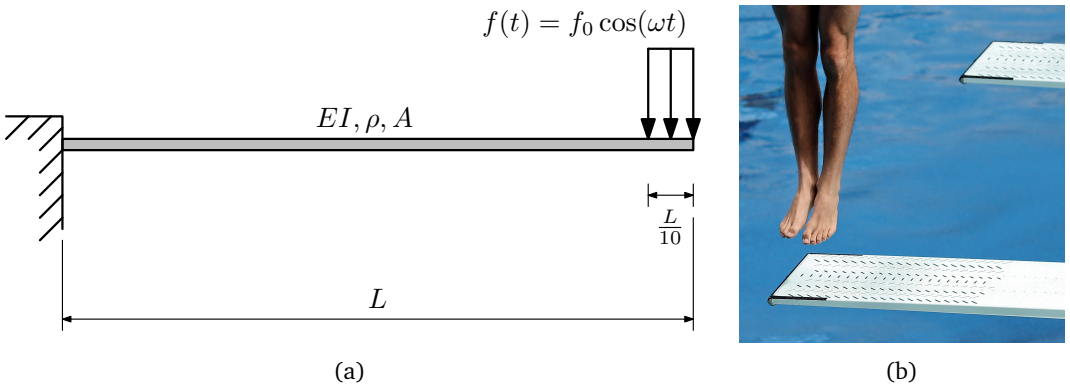
Finally, substituting the known values we get

$$A_1 = \frac{20 \times 1 \times 10^{-3} \times 1.4}{9 \times 1818.8 \times 8500 \times 7 \times 10^{-3} \times 0.8 \times 10^{-3} \times 36 \times 10^{-3}} = 0.998 \times 10^{-3} \approx 1.0 \text{ mm}$$

Question 7.7

The cantilever beam shown in Figure 4a is a model of the 3 meter Olympic diving board in Figure 4b[1]. The board has density ρ, cross section area A, length L and bending stiffness EI. A pulsating distributed force $f(t) = f_0 \cos(\omega t)$ acts on the board as shown.

a) Suggest two alternative normalised shape functions for the system.
b) Select one of the shape functions from part (a) and derive the equation of motion for the system.
c) Consider that $L = 3$ m, the width and thickness of the board are $b = 500$ mm and $h = 30$ mm, respectively, and determine if pine wood ($\rho = 500$ kg/m^3, $E = 9$ GPa) is a good material to ensure that the natural frequency of the board is lower than 10 rad/s.



(a) (b)

Figure 4: Model of the 3 meter Olympic diving board for Question 7.7.

Example solution

For this example solution, the origin of the cartesian system Oxy is positioned at the left end of the beam. The deflection of the beam can be described by the following generic expression:

$$w(x,t) = q(t)\varphi(x)$$

The shape functions must satisfy the boundary conditions at $x = 0$ and be normalised, that is, $\varphi(x) \in [-1, +1]$. A possible set of functions that satisfies these conditions is

$$\varphi(x) = \left(\frac{x}{L}\right)^2 \quad \text{or} \quad \varphi(x) = 1 - \cos\left(\frac{\pi x}{2L}\right)$$

The expression for the kinetic energy of the beam is

$$T = \frac{\rho A}{2} \int_0^L \left[\frac{\partial w(x,t)}{\partial t}\right]^2 \mathrm{d}x$$

After replacing the deflection $w(x,t)$ and the corresponding time derivatives, the kinetic energy becomes

$$T = \frac{1}{2}\left(\frac{\rho A L}{5}\right)\dot{q}^2 = \frac{1}{2}\bar{M}\dot{q}^2$$

Neglecting the self-weight of the beam, and thus the gravitational energy terms, the expression for the potential energy of the beam becomes

$$V = \frac{1}{2} EI \int_0^L \left[\frac{\partial^2 w(x,t)}{\partial x^2}\right]^2 \mathrm{d}x$$

After replacing the deflection and the corresponding derivatives, the potential energy becomes

$$V = \frac{1}{2}\left(\frac{4EI}{L^3}\right) q^2 = \frac{1}{2}\bar{K}q^2$$

The natural frequency of the diving board is then

$$\omega^2 = \frac{\bar{K}}{\bar{M}} = 20\frac{EI}{mL^3}$$

[1] Photo licensed under the Creative Commons Attribution 2.0 Generic License (CC BY 2.0).

As the pulsating force is defined by $f(t) = f_0 \cos(wt)$ for $\frac{9L}{10} < x < L$, the generalised force acting on the system is

$$Q = \int_{\frac{9L}{10}}^{L} f(t)\varphi(x)\mathrm{d}x$$

Introducing the shape function and external force the generalised force becomes

$$Q = \int_{\frac{9L}{10}}^{L} f_0 \cos(wt) \left(\frac{x}{L}\right)^2 \mathrm{d}x = \frac{f_0 \cos(wt)}{3L^2}\left[L^3 - \left(L - \frac{L}{10}\right)^3\right] = 0.09 f_0 L \cos(wt)$$

Applying Lagrange's method the equation of motion of the diving board is

$$\frac{1}{5}\rho A L \ddot{q} + \frac{4EI}{L^3} q = 0.09 f_0 L \cos(wt)$$

For the natural frequency of the diving board to be lower than 10 rad/s it is necessary that

$$\omega < 10 \text{ rad/s} \qquad \Rightarrow \qquad 20\frac{EI}{mL^3} < 100 \qquad \Rightarrow \qquad EI < 5mL^3$$

The mass of the board is

$$m = \rho A L = 500 \times 0.5 \times 0.03 \times 3 = 22.5 \text{ kg}$$

The moment of inertia is

$$I = \frac{bh^3}{12} = \frac{0.5 \times 0.03^3}{12} = 10^{-6} \text{ m}^4$$

To ensure that the natural frequency of the diving board is lower than 10 rad/s

$$EI < 5 \times 22.5 \times 3^3 \qquad \Rightarrow \qquad E < \frac{5 \times 22.5 \times 3^3}{10^{-6}} = 3 \text{ GPa}$$

In conclusion, pine wood is not a good material for the diving board as its elastic modulus is not lower than 3 GPa.

Question 7.8

Consider a simply-supported homogeneous beam of length L, density ρ, elastic modulus E, cross-section area A and second moment of area I. The structure is impacted by a projectile of mass m with velocity v_0 at location x_0 from one end of the beam.

a) Propose a trigonometric shape function for the first natural frequency of the beam.
b) Show that the first natural frequency is independent of the normalisation of the shape function.
c) Investigate if the modified frequency $\bar{\omega}$ and amplitude of vibration A depend on the normalisation of the shape function.

Example solution

Assuming the left end of the beam to be at $x = 0$, a possible trigonometric shape function for the first natural frequency of the beam is

$$\varphi(x) = \sin\left(\frac{\pi x}{L}\right)$$

This function is normalised as it maximum is

$$\varphi_{\max} = \varphi\left(\frac{L}{2}\right) = \sin\left(\frac{\pi L}{2L}\right) = 1$$

The equivalent non-normalised shape function is

$$\varphi^\star(x) = B\sin\left(\frac{\pi x}{L}\right) \quad \text{with} \quad B \neq 1$$

The natural frequency of the beam can be determined with Rayleigh's method, with

$$\omega^2 = \int_0^L EI\left(\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2}\right)^2 \mathrm{d}x \Big/ \int_0^L \rho A\varphi^2 \mathrm{d}x = \frac{EI}{\rho A}\int_0^L \left(\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2}\right)^2 \mathrm{d}x \Big/ \int_0^L \varphi^2 \mathrm{d}x$$

Using the normalised shape function $\varphi(x)$ the frequency then becomes

$$\omega^2 = \frac{EI}{\rho A}\left(\frac{\pi}{L}\right)^4 \int_0^L \sin^2\left(\frac{\pi x}{L}\right)\mathrm{d}x \Big/ \int_0^L \sin^2\left(\frac{\pi x}{L}\right)\mathrm{d}x \quad \text{and} \quad \omega = \left(\frac{\pi}{L}\right)^2\sqrt{\frac{EI}{\rho A}}$$

Following the same approach for the non-normalised shape function $\varphi^\star(x)$ we obtain

$$\omega^{\star 2} = \frac{EI}{\rho A}\left(\frac{\pi}{L}\right)^4 \int_0^L B^2\sin^2\left(\frac{\pi x}{L}\right)\mathrm{d}x \Big/ \int_0^L B^2\sin^2\left(\frac{\pi x}{L}\right)\mathrm{d}x$$

The constant B^2 cancels leading to

$$\omega^\star = \left(\frac{\pi}{L}\right)^2\sqrt{\frac{EI}{\rho A}} = \omega$$

showing that the (first) natural frequency is independent of the normalisation of the shape function. Upon impact, the modified frequency is

$$\bar{\omega}^2 = \omega^2 \Big/ \left(1 + \frac{m\varphi(x_0)}{\rho A\int \varphi^2 \mathrm{d}x}\right)$$

As the shape function is squared in the denominator but not in the numerator, using a non-normalised function will lead to a factor of B in the denominator that is not cancelled, that is

$$\bar{\omega}^{\star 2} = \omega^2 \Big/ \left(1 + \frac{mB\varphi(x_0)}{\rho A\int B^2\varphi^2 \mathrm{d}x}\right) = \omega^2 \Big/ \left(1 + \frac{m\varphi(x_0)}{\rho A\int B\varphi^2 \mathrm{d}x}\right) \neq \bar{\omega}^2$$

A similar approach can be used to investigate the post-impact amplitude of vibration,

$$A^\star = \frac{mv_0 B\varphi(x_0)}{\bar{\omega}\rho A\int B^2\varphi^2 \mathrm{d}x} = \frac{mv_0\varphi(x_0)}{\bar{\omega}\rho A\int B\varphi^2 \mathrm{d}x} \neq A$$

Consequently, both the modified frequency and the amplitude depend on the normalisation of the shape function. Note that in the amplitude, the modified frequency in the denominator also contributes to this.